Particle Physics 2: Quantum Chromodynamics

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2 The QCD Lagrangian and its Symmetry Structure

After this introduction to the basic motivation to QCD as a theory of the strong interactions, let us discuss its Lagrangian and its basic properties, in particular its symmetries, and what are the consequences of these for phenomenology.

Learning Goals of the Lecture

- Derive the QCD Lagrangian from symmetry principles.
- Determine the qualitative implications of this Lagrangian for QCD phenomenology.
- Distinguish between local and global symmetries.
- Derive the renormalisation group equations that govern the running of the QCD coupling.

2.1 The QED Lagrangian revisited

In previous courses you have encountered the Lagrangian of Quantum Electrodynamics (QED), which has the following structure:

$$\mathcal{L}_{\rm QED} = \sum_f \overline{\psi}_f \left(i\gamma^\mu D_\mu - m_f\right)\psi_f - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}\,, \quad (2.1)$$

where the sum runs over all the fermions present in the theory. In the following for simplicity we consider a single fermion of mass m. In this Lagrangian, the covariant derivative is defined as

$$D_\mu \equiv \partial_\mu + ieA_\mu\,, \quad (2.2)$$

with e being the electric charge of this fermion, and the field strength tensor is the usual definition from classical electrodynamics

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu\,, \quad (2.3)$$

with A_μ indicating the photon field.

The basic property of the QED Lagrangian is the invariance over **local U(1) transformations**, where U(1) is the abelian rotation group. This is know as the **gauge symmetry** of the theory. Under such rotation, fermion fields transform as

$$\psi(x) \to \psi'(x) = e^{i\phi(x)}\psi(x)\,, \tag{2.4}$$

and we know from classical electrodynamics that the electromagnetic fields are invariant if we modify the vector potential can always be modified by an additive total derivative of the form

$$A^\mu \to A'^\mu = A^\mu + \partial^\mu \chi(x)\,, \tag{2.5}$$

with $\chi(x)$ being an arbitrary scalar function. It is easy to see that the field strength tensor is invariant under gauge transformations, namely

$$F_{\mu\nu} \to F'_{\mu\nu} \to F_{\mu\nu}\,. \tag{2.6}$$

The QED Lagrangian will be gauge invariant if the covariant derivative transforms in the same way as the fermion field

$$D_\mu\psi \to D'_\mu\psi' = e^{i\phi(x)}D_\mu\psi\,, \tag{2.7}$$

which as can be seen is achieved when we choose the gauge transformation of the four-vector potential to be the following:

$$\chi(x) = -\frac{\phi(x)}{e}\,, \tag{2.8}$$

To see this, note that

$$
\begin{aligned}
D_\mu\psi &= (\partial_\mu + ieA_\mu)\,\psi\,, &(2.9)\\
D'_\mu\psi' &= (\partial_\mu + ieA_\mu - i(\partial_\mu\phi))\,e^{i\phi(x)}\psi\,, &(2.10)\\
D'_\mu\psi' &= e^{i\phi(x)}\,(\partial_\mu + ieA_\mu)\,\psi\,, &(2.11)
\end{aligned}
$$

which ensures the sought-for gauge invariance of the QED Lagrangian.

Discussion #2.1

In addition to gauge symmetry, can you indicate other (local or global, continuous or discrete) symmetries which are **satisfied by the QED Lagrangian?** And why identifying symmetries is so important to understand the physical properties of a theory?

2.2 The SU(3) gauge group and color singlet states

As we have discussed in the previous chapter, as compared to QED the main difference in the QCD case is the existence of **new internal quantum number**, color, which, in the language of gauge theory, leads to an invariance under a different non-abelian group. QED is an abelian gauge theory, where the relevant gauge group is the abelian U(1). As opposed to it, the gauge group of QCD is SU(3), the group of specially unitary transformations of degree $n = 3$. The reason is that quarks can have $N_c = 3$ **different types of color charges**, as opposed to QED where only a single type of electric charge exists. This group is defined by all $n \times n$ unitary matrices

$$U^T U = \mathbb{1}\,, \tag{2.12}$$

which have determinant equal to 1, det(U) = 1. The fact that the gauge invariance of QCD is under a non-abelian group has important consequences, as we will see now.

Constructing the QCD Lagrangian requires, in the same way as for QED, identifying terms which are **invariant** under a local SU(3) gauge transformation. In the fundamental representation, a suitable choice

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} ,$$

$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \quad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} ,$$

$$\lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \quad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} .$$

Figure 2.1. The Gell-Mann matrices λ^a for the fundamental representation of SU(3), whose commutation relations are given by the group's Lie algebra.

of generators for SU(3) are the Gell-Mann matrices, which are hermitian and traceless,

$$t^a \equiv \frac{1}{2}\lambda^a \,, \qquad a = 1, \ldots, 8 \,, \tag{2.13}$$

and which obey the commutation relations of the group's Lie algebra

$$\left[t^a, t^b\right] = i f^{abc} t^c \,, \tag{2.14}$$

with f^{abc} the structure constants of SU(3), namely

$$f_{123} = 1 \tag{2.15}$$

$$f_{147} = -f_{156} = f_{246} = f_{257} = f_{345} = -f_{367} = \frac{1}{2} \tag{2.16}$$

$$f_{458} = f_{678} = \frac{\sqrt{3}}{2} \tag{2.17}$$

and the corresponding permutations. All other structure constants are zero. The eight Gell-Mann matrices for the fundamental representation of SU(3) are summarized in Fig. 2.1.
Let us define the quark fermion wave-function as follows:

$$\psi^{(f)\leftarrow \text{flavor}}_{i \leftarrow \text{color}}(x) \,, \tag{2.18}$$

which now has two indices: a flavor index f (like in QED) and a color index i (a genuine new feature of QCD). The color index was take any value up to N_c, the number of colors in the theory. This color-charged fermion field will be the building-block of the QCD Lagrangian. Quarks transform under color in the fundamental representation of SU(3), which means:

$$\psi^{(f)}_i \to \psi^{(f)'}_i = U_{ij}(x)\psi^{(f)}_j \tag{2.19}$$

which note that it acts only on the color degrees, and leaves the flavor degrees of freedom untouched. In other words, an SU(3) gauge transformation is diagonal in flavor space. The SU(3) transformation is given by

$$U_{ij}(x) = \exp\left(i\theta^a(x)\, t^a_{ij}\right) \,, \tag{2.20}$$

in terms of the Gell-Mann matrices given in Eq. (2.13). One can compare the SU(3) gauge transformation relevant for QCD, Eq. (2.19), with its counterpart in QED, Eq. (2.4). The main difference is that an SU(3) transformation is a **rotation in color space** which depends on the space-time point x, while in the case of QED the gauge transformation is a local overall rescaling of the fermion field without a rotation.

Color singlet states in QCD

All experimentally observed strongly-interesting particles are found to be color-singlet. In other words, a net color charge is not allowed for hadrons. As we will see soon, the reason for this is that QCD is a **confining theory**: the strength of the interactions between quarks grows with their separation, to the point than an infinite energy would be necessary to asymptotically separate a quark from an antiquark.

Since a net color charge is not allowed for hadrons, the total color charge of physical states must be zero. This imposes some restrictions on the kind of quark and antiquark combinations which can form physical hadrons. To see this, consider the the wave function of a **quark-antiquark bound state** (known as a meson), constructed as

$$\sum_{i=1}^{N_c} \psi_i^{*(f)} \psi_i^{(f')} \,, \tag{2.21}$$

which can be shown to be **invariant under a a SU(3) transformation** of the form indicated in Eq. (2.19), since:

$$\sum_{i=1}^{N_c} \psi_i^{*(f)} \psi_i^{(f')} \to \sum_{i=1}^{N_c} \left(\psi_i^{*(f)}\right)' \left(\psi_i^{(f')}\right)' =$$
$$\sum_{i}^{N_c} \left(\sum_{j}^{N_c} U_{ij}^{*} \psi_j^{*(f)}\right) \left(\sum_{k}^{N_c} U_{ik} \psi_k^{(f)}\right) = \sum_{kj} \left(\sum_{i} U_{ji}^{T} U_{ik}\right) \psi_j^{*(f)} \psi_k^{(f')} = \sum_{k=1}^{N_c} \psi_k^{*(f)} \psi_k^{(f')} \,, \tag{2.22}$$

using the unitarity properties of the SU(3) matrices given by Eq. (2.12). We conclude that a quark with flavor index f' and an antiquark with flavor index f can form a state which is invariant under SU(3) gauge transformation, and hence it is a **color singlet state** and physically allowed.

Discussion #2.2

Can you indicate some possible combinations of quarks and antiquarks which satisfy the structure of Eq. (2.21)? Are here the flavour indices f and f' arbitrary? And can f correspond to a top quark?

The above discussion indicates that quark-antiquark states satisfy the zero net color condition expected for hadrons. This property must also hold for particles such as protons and neutrons, which are composed by three quarks. This can be achieved if the wave function of three-quark states is constructed as follows, using the antisymmetric tensor:

$$\sum_{ijk}^{N_c} \epsilon^{ijk} \psi_i^{(f)} \psi_j^{(f')} \psi_k^{(f'')} \,, \tag{2.23}$$

which can be shown to be color singlet using the relation

$$\sum_{ijk} \epsilon^{ijk} U_{ii'} U_{jj'} U_{kk'} = [\det \mathrm{U}] \; \epsilon^{i'j'k'} \,. \tag{2.24}$$

and the fact that the SU(3) matrices are unitary (as above). The same considerations apply for a wave function composed by three antiquarks. Hadrons composed by three (anti)quarks are known as **baryons** (antibaryons) and as expected can exist in a color singlet configuration.

Discussion #2.3

Can bound states composed by more than three quarks exist in Nature given the symmetries of QCD? For instance, do tetraquarks or pentaquarks exist?

2.3 The QCD Lagrangian and its symmetries

Following this discussion, we are ready to present the QCD Lagrangian and compare its main features with those present in its QED counterpart Eq. (2.1). As in the case of QED, the guiding principle to construct the QCD Lagrangian is its invariance under local SU(3) transformations, in addition to invariance under the other symmetries of the theory such as Lorentz transformations. The QCD Lagrangian is given by:

$$\mathcal{L}_{\rm QCD} = \sum_{f=1}^{n_f} \overline{\psi}_i^{(f)} \left(i\gamma_\mu D^\mu_{ij} - m_f \delta_{ij} \right) \psi_j^{(f)} - \frac{1}{4} F^a_{\mu\nu} F^{\mu\nu}_a \,, \tag{2.25}$$

where a is a color index that runs from 1 to $N_c - 1 = 8$, and we are assuming n_f massive fermions. As in the case of QED, a possible mass term for the gluon field of the form $m^2 A^\mu_a A_{\mu,a}$ would be forbidden by gauge invariance. Note that now the **covariant derivative carries color indices** ij, since it implements a rotation in color space:

$$D^\mu_{ij} = \partial^\mu \delta_{ij} + i g_s t^a_{ij} A^\mu_a \,, \tag{2.26}$$

in terms of the Gell-Mann matrices of SU(3). Here we denote as g_s as the coupling constant of the strong interactions. The field strength tensor is given by

$$F^a_{\mu\nu} \equiv \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g_s f_{abc} A^b_\mu A^c_\nu \tag{2.27}$$

with f_{abc} the structure constants of SU(3) mentioned above. The gauge transformation property of the gluon field is given by

$$t^a A^\mu_a \to t^a A^{'\mu}_a = U(x) t^a A^\mu_a U^{-1}(x) + \frac{1}{g_s} \left(\partial^\mu U(x) \right) U^{-1}(x) \,, \tag{2.28}$$

which can be shown to lead to a covariant derivative which transform as the quark field itself

$$D^\mu_{ij}\psi_j \to \left(D^\mu_{ij}\psi_j \right)' = U_{ik}(x) D^\mu_{kj}\psi_j \,, \tag{2.29}$$

exactly as in the case of QED.

Discussion #2.4

The gauge transformation of the gluon field under color, Eq. (2.28), seems to be quite different from the one in QED, Eq. (2.5). What is the main reason of the difference? What happens if the gauge group is Abelian?

An important property in order to prove that the gauge sector of the QCD Lagrangian is gauge invariant is the following

$$[D_\mu, D_\nu] = i g_s \, t^a \, F^a_{\mu\nu} \,, \tag{2.30}$$

which can be derived from the definition of the covariant derivative acting on a fermion field. An important difference between QED and QCD is that the field-strength tensor itself is **not** gauge invariant. This can be seen as follows: the transformation law of the field strength tensor under SU(3) transformations will be given by

$$t^a F^a_{\mu\nu} \to t^a F^{'a}_{\mu\nu} = U(x)\, t^a F^a_{\mu\nu}\, U^{-1}(x) \qquad (2.31)$$

which is hence not invariant under gauge transformations. However, the product $F^a_{\mu\nu} F^{\mu\nu}_a$ is invariant under SU(3) transformations, and hence the gluon sector of the Lagrangian is also so (else this term could not have been present in the QCD Lagrangian Eq. (2.25). To show this, we can use the following property of SU(3):

$$\text{Tr}\left[t^a t^b\right] = \frac{1}{2}\delta^{ab}\,, \qquad (2.32)$$

and therefore we can have the following

$$-\frac{1}{4}F^a_{\mu\nu}F^{\mu\nu}_a = -\frac{1}{2}F^a_{\mu\nu}\, F^{\mu\nu,b}\text{Tr}\left[t^a t^b\right] = -\frac{1}{2}\text{Tr}\left[F^a_{\mu\nu}\, t^a\, F^{\mu\nu,b}\, t^b\right]\,. \qquad (2.33)$$

which is invariant under the SU(3) transformation of Eq. (2.31) using to the cyclic properties of the trace. Therefore, the purely gluonic piece of the QCD Lagrangian is indeed gauge invariant, even if the individual field-strength tensor is not.

2.4 Chiral symmetry in QCD

Approximate symmetries in QCD

Protons and neutrons are found to behave almost in an identical manner under the strong interaction, to begin with by having a very similar mass. The same feature holds for other hadrons, for example neutral and charged pions, despite their different flavor composition, they behave in the same manner under the strong force. The reason is chiral symmetry, which is an **approximate symmetry** (as opposed to an exact one like gauge invariance) of the QCD Lagrangian.

To be more precise, strong interactions are found experimentally to behave very similar for particles, like protons and neutrons, that arise in the same **isospin multiplet**. Isospin is an approximate **global SU(2) symmetry** (and not local, such as gauge transformation) which relates the up and the down content of hadrons. Isospin (approximate) symmetry arises from the fact that the masses of the two lighter quarks are close to each other, $m_u \simeq m_d$.

Discussion #2.5

What is the underlying reason explaining $m_u \simeq m_d$ and hence chiral symmetry in QCD?

In the quark model, the isospin content of hadrons is defined as follows

$$I_3 = \frac{1}{2}\left[(n_u - n_{\bar{u}}) - (n_d - n_{\bar{d}})\right]\,, \qquad (2.34)$$

with $n_{u(d)}$ indicating the number of up (down) quarks contained in a given hadron. Therefore we have that I_3 is 1/2 for protons and $-1/2$ for neutrons, since these are members of the same isospin multiplet. Formally, an isospin transformation acts on the quark field as a unitary (global) SU(2) matrix

$$\psi_i^{(f)} \to \sum_{f'=1}^{2} U^{ff'} \psi_i^{(f')} \,, \tag{2.35}$$

This looks superficially similar to the color SU(3) transformation of Eq. (2.19), but they are completely different: this is a global transformation (not a local one as in the case of gauge symmetries) and it leaves the color indices unchanged.

To study under which conditions the fermion sector of the QCD Lagrangian is invariant under isospin transformations, we can separate the up and down quarks from all other fermionic terms:

$$\begin{aligned} \mathcal{L}_{\rm QCD} &= \overline{\psi}_i^{(u)} \left(i\gamma_\mu D^\mu_{ij} - m_u \delta_{ij} \right) \psi_j^{(u)} + \overline{\psi}_i^{(d)} \left(i\gamma_\mu D^\mu_{ij} - m_d \delta_{ij} \right) \psi_j^{(d)} \\ &+ \sum_{f, f \neq u,d} \overline{\psi}_i^{(f)} \left(i\gamma_\mu D^\mu_{ij} - m_f \delta_{ij} \right) \psi_j^{(f)} - \frac{1}{4} F^a_{\mu\nu} F_a^{\mu\nu} \,. \end{aligned} \tag{2.36}$$

Restricting the fermion sector to only up and down quarks, the isospin transformed QCD Lagrangian reads

$$\sum_{f',f''} \sum_f \left(U^T_{f'f} U_{ff''} \right) \overline{\psi}_i^{(f')} \left(i\gamma_\mu D^\mu_{ij} - m_f \delta_{ij} \right) \psi_j^{(f'')} \,. \tag{2.37}$$

Since SU(2) is an unitary group, if $m_u = m_d$, including the case $m_d = m_u = 0$, we can use that

$$\sum_f U^T_{f'f} U_{ff''} = \mathbb{1}_{f'f''} \,, \tag{2.38}$$

and this this case the Lagrangian will be exactly invariant under chiral symmetry.

On the origin of hadron massess. Experimentally, we know that the up and down quark masses are much smaller than the typical scale of the QCD interactions

$$m_{u,d} \ll \Lambda_s \,. \tag{2.39}$$

In the limit of quark masses vanishing, we can separate left-handed and right-handed fermion chiralities and the Lagrangian is separately invariant for the two components

$$\psi = \psi_R + \psi_L \,, \qquad \psi_L = \frac{1}{2}(1-\gamma_5)\psi \,, \qquad \psi_R = \frac{1}{2}(1+\gamma_5)\psi \,, \tag{2.40}$$

$$\sum_f \left(\overline{\psi}_R^{(f)} \left(i\gamma_\mu D^\mu \right) \psi_R^{(f)} + \overline{\psi}_L^{(f)} \left(i\gamma_\mu D^\mu \right) \psi_L^{(f)} \right) \tag{2.41}$$

As a result of this invariance, **chirality will be a conserved quantum number** for massless fermions. In this case, the QCD Lagrangian would be chirally invariant. This means in particular that **hadrons composed by light quarks would be massless**, since a mass term violates chirality conservation by mixing left-handed and right-handed chiralities.

However, we know that QCD is not chirally invariant (even if up and down quarks were exactly massless), and this arises because of the **spontaneous breaking of chiral symmetry**. Spontaneous breaking of a symmetry occurs when the symmetry group of the solutions of a theory is dynamically generated to be less than the symmetry of the original Lagrangian, as for example in the case of the Higgs mechanism. In the case of QCD, it is known that the vacuum has a non-zero expectation value of the light quark operator

$$\langle 0|\bar{q}q|0\rangle = \langle 0|\bar{u}u + \bar{d}d|0\rangle \simeq (250 \text{ MeV})^3 \tag{2.42}$$

which breaks chiral symmetry and is responsible for most of the hadron masses.

Discussion #2.6

Which is approximately the contribution of the Lagrangian quark masses to the **observed hadrons masses**? Can we then conclude that the Higgs mechanism is responsible for giving most of the mass to visible matter?

2.5 Color flows and Feynman rules in QCD

From the structure of the QCD Lagrangian in Eq. (2.25), by means of the quantum field theory machinery we can evaluate the Feynman rules necessary to evaluate scattering amplitudes and decay rates in QCD.

Color flows. We note that the various contractions of color indices in the QCD Lagrangian can be interpreted as **color flows** between the different types of fields. In the specific example of Fig. 2.2, an incoming quark with red color emits a gluon and is transformed into an outgoing quark with blue color. In this picture, gluons carry both color and anti-color: they change the color charge of quarks and of other gluons. The allowed color flows are restricted by SU(3) symmetry, as can be seen in the example of Fig. 2.2 with the Gell-Man matrix.

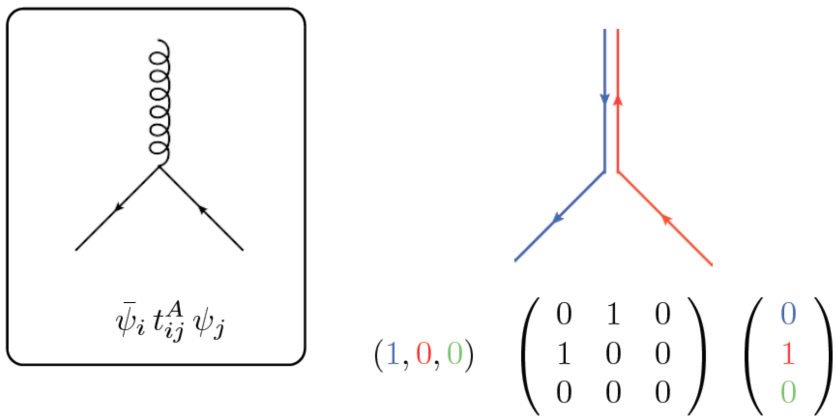

Figure 2.2. The emission of a gluon from a quark line can be interpreted in terms of color flow along the quark and gluon lines. This QCD $q\bar{q}g$ vertex induces a rotation in color space, and the gluon line can be represented as a color-anticolor pair flow.

While the picture shown in Fig. 2.2 is intuitive, we should mention that it only holds in the the so-called **large-N_c limit** of QCD, where we take the limit $N_c \to \infty$. In this limit we can replace

$$(A_\mu)^a \to (A_\mu)^i_j \,, \tag{2.43}$$

where a is an index in the adjoint representation and i, j are indices in the fundamental representation of SU(3). While $N_c = 3$ does not seem to be a good approximation for $N_c \to \infty$, actually the large-N_c limit of QCD works quite well and benefits from many phenomenological applications.

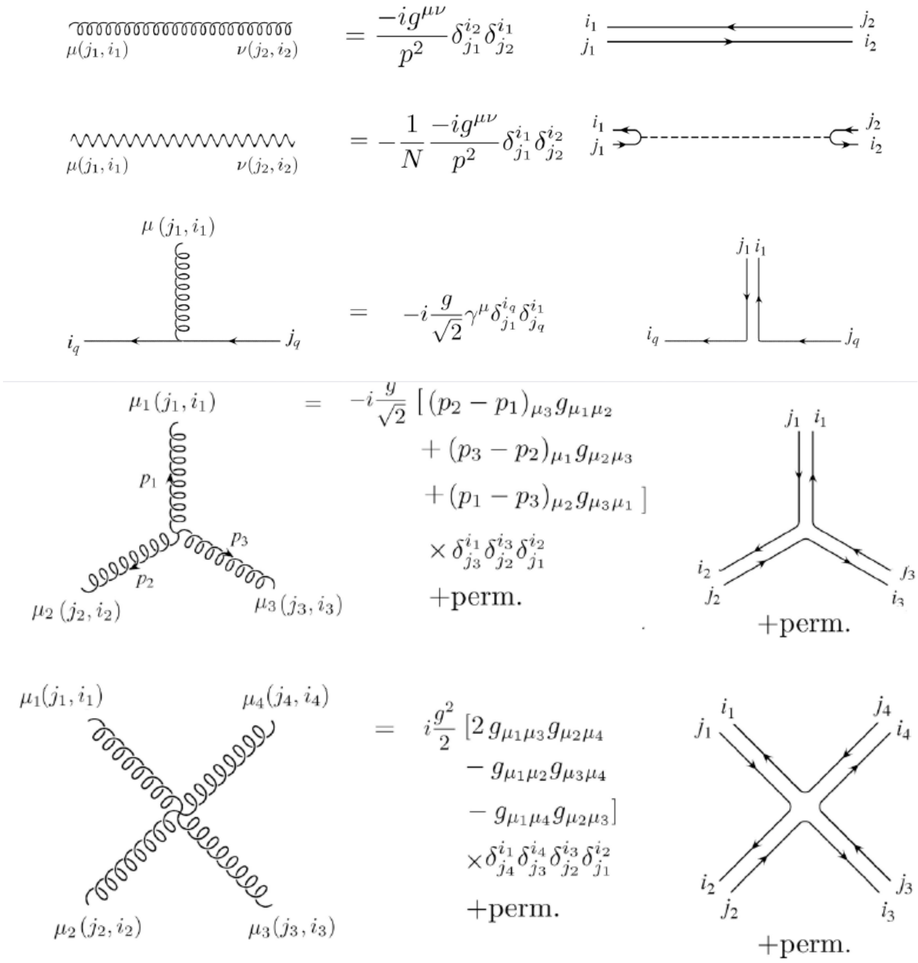

Figure 2.3. The Feynmann rules of QCD, for the propagation and interaction of quarks and gluons. In the right part of the plot, we show how the Feynman rules can be also interpreted in terms of color flow between quarks and gluons, which as mentioned above only hold in the $N_c \to \infty$ limit.

Feynman rules. Following the same techniques as in the case of QED, it is possible to determine the Feynman rules that QCD must obey. These Feynman rules are summarized in Fig. 2.3. In the left panel of the plot, we show the various propagators and vertices that are allowed in QCD, with the corresponding Feynman rules. In the right part of the panel, we show how the Feynman rules can be also interpreted in terms of color flow between quarks and gluons, which as mentioned above only hold in the $N_c \to \infty$ limit.

Note in particular the presence of **three-gluon and four-gluon vertices**, which are absent in QED: this is a genuine new feature of QCD, which arises from the non-abelian nature of $SU(3)$, and that has important consequences, like asymptotic freedom. In the three-gluon diagram of Fig. 2.3, p_i are the four momenta of each of the gluons, assuming they are incoming. When computing Feynman diagrams, all possible permutations of the three and four-gluon vertices need to be included.

In order to understand intuitively where the three- and four-gluon vertices come from, one can do the following. Let's expand the purely gluonic term of the QCD Lagrangian (field strength product), keeping only the terms proportional to the strong coupling constant. Starting from the expression for the field strength tensor in Eq. (2.27), we find the following terms:

$$
\begin{aligned}
F_a^{\mu\nu} F^a_{\mu\nu} \quad &\to \quad \ldots + g_s^2 \, f_{abc} \, f_{ade} \, A^{\mu,b} \, A^{\nu,c} \, A^d_\mu \, A^e_\nu \\
&- \quad g_s \, f_{abc} \, A^{\mu,b} \, A^{\nu,c} \left[\partial_\mu \, A^a_\nu - \partial_\nu \, A^a_\mu \right] - g_s \, f_{abc} \, A^b_\mu \, A^c_\nu \left[\partial^\mu A^{\nu,a} - \partial^\nu A^{\mu,a} \right] ,
\end{aligned} \tag{2.44}
$$

which lead to the vertices shown in Fig. 2.3. The proportionality to p_i in the three-gluon vertex arises the derivatives in the second half of the equation above (since in Fourier space a derivative is represented by a momentum insertion).

In Fig. 2.3, right panel, we also show how the Feynman rules can be understood in terms of color flow in the $N_c \to \infty$ limit. In all these diagrams, we can see that color is conserved, since no color lines are interrupted or discontinued.

The Feynman rules summarized in Fig. 2.3 represent the building block of the perturbative QCD calculations which we will carry out during this course.

2.6 Asymptotic freedom and confinement in QCD

One crucial property of QCD is that of **asymptotic freedom**: the strength of the QCD interaction $\alpha_s(Q^2) = g_s^2(Q^2)/4\pi$ decreases when the momentum transfers Q^2 involved in the interaction increase. This property allows one to use the basic tools of perturbative QFT to deal with hard QCD interactions, in the same way as with the usual QED calculations. On the other hand, it also implies that at low scales QCD becomes non-perturbative, since the QCD coupling diverges, and the perturbative interpretation is not valid anymore.

QCD as a QED-like QFT?

The fact that hadrons interact so strongly made researchers in the 50s and 60s doubt whether a QFT-like formulation of the strong nuclear force would be appropriate. The formulation of QCD in the QFT framework became only possible once it was shown that the same theory can behave **both QED-like** (weakly coupled) at small distances (large momentum transfers) as well as **non-perturbative** (strongly coupled) at large distances (small momentum transfers).

Discussion #2.7

When we say talk about "large" or "small" momentum transfers in QCD we need to specify the **characteristic mass scale** of QCD taken as reference. Which could be the options here?

While asymptotic freedom in QCD suggests confinement, since the strength of the interaction between two quarks would increase the more we try to separate them (larger distances corresponds to smaller energy scales), **confinement has never been formally derived** from the QCD Lagrangian, and only numerical evidence from lattice simulations has been obtained.

The running of the coupling constants in Quantum Field Theories like QED or QCD can be understood from **renormalization group evolution** arguments. To show this, let us consider a given observable in a generic QFT with (bare) dimensionless coupling constant α, and a number of kinematical invariants $s_1 \ldots s_n$. In quantum field theories we encounter **ultraviolet divergences**, which in renormalizable theories can be removed by a suitable redefinition of the couplings and fields. If we denote by M the UV cut-off, after renormalization the observable G will read

$$
G = G(\alpha, \, M, \, s_1, \ldots, s_n) \, , \tag{2.45}
$$

with the cut-off regulating the previous UV divergence. The dependence on the cut-off scale M can be eliminated by a suitable redefinition of the bare coupling (which is scale independent) in terms of the renormalized coupling (which now will depend on the scale), order by order in the perturbative expansion

$$\alpha_{\rm ren}(\mu) = \alpha + \sum_{\ell=2} c_\ell \left(M/\mu\right) \alpha^\ell \,, \tag{2.46}$$

where the coefficients c_ℓ are dimensionless, and the arbitrary scale μ has been introduced for dimensional reasons. This scale μ is known as the **renormalization scale**, and arises also if one regulates UV divergences using e.g. dimensional regularization.

Any physical quantity in the theory can be expressed in terms of the renormalized coupling, the renormalization scale μ and other kinematical invariants, without the need anymore of the UV cutoff:

$$G\left(\alpha\left(\alpha_{\rm ren}, M/\mu\right), \, M, \, s_1, \ldots, s_n\right) = \widetilde{G}\left(\alpha_{\rm ren}, \mu, s_1, \ldots, s_n\right) \,, \tag{2.47}$$

In renormalizable QFTs, a single redefinition of the coupling makes finite **all** physical observables. The price to pay is the dependence of our results on the arbitrary renormalization scale μ, which could be eliminated only by computing the theory to all perturbative orders.

The renormalization group equations (RGE) arise from the condition that if we vary $\alpha_{\rm ren}$ and μ, keeping α and M fixed, cross-sections should be invariant, since before renormalization, **physical observables depend only on M and α**. This leads to the conditions

$$\frac{\partial \widetilde{G}\left(\alpha_{\rm ren}, \mu, s_1, \ldots, s_n\right)}{\partial \alpha_{\rm ren}} d\alpha_{\rm ren} + \frac{\partial \widetilde{G}\left(\alpha_{\rm ren}, \mu, s_1, \ldots, s_n\right)}{\partial \mu} d\mu = 0 \tag{2.48}$$

$$\frac{\partial \alpha\left(\alpha_{\rm ren}, M/\mu\right)}{\partial \alpha_{\rm ren}} d\alpha_{\rm ren} + \frac{\partial \alpha\left(\alpha_{\rm ren}, M/\mu\right)}{\partial \mu^2} d\mu^2 = 0 \tag{2.49}$$

and from these expressions it is possible to determine that the dependence of $\alpha_{\rm ren}$ with the scale μ:

$$\mu^2 \frac{d\alpha_{\rm ren}}{d\mu^2} = \beta(\alpha_{\rm ren}) \,, \tag{2.50}$$

given that

$$\mu^2 \frac{d\alpha_{\rm ren}}{d\mu^2} = -\frac{\mu^2 \partial \alpha\left(\alpha_{\rm ren}, M/\mu\right)/\partial \mu^2}{\partial \alpha\left(\alpha_{\rm ren}, M/\mu\right)/\partial \alpha_{\rm ren}} = -\frac{\mu^2 \partial \widetilde{G}\left(\alpha_{\rm ren}, \mu, s_1, \ldots, s_n\right)/\partial \mu^2}{\partial \widetilde{G}\left(\alpha_{\rm ren}, \mu, s_1, \ldots, s_n\right)/\partial \alpha_{\rm ren}} \tag{2.51}$$

and therefore the functional dependence of the renormalized coupling with the scale μ is fixed by the following conditions:

- It cannot depend on M, since RHS does not.
- It cannot depend on the kinematical invariants, since LHS does not.
- It cannot depend on μ, since we don't have other dimensionful variables available and the coupling is dimensionless.

At the first non-trivial order, the RGE equation Eq. (2.50) reads

$$\frac{d}{d\,\log \mu^2} \alpha_{\rm ren}(\mu^2) = -b_0 \alpha^2_{\rm ren}(\mu) \,, \tag{2.52}$$

which can be solved to give:

$$\alpha_{\rm ren}(\mu^2) = \frac{\alpha_{\rm ref}(\mu_0^2)}{1 + \alpha_{\rm ref}(\mu_0^2) b_0 \ln \frac{\mu^2}{\mu_0^2}} \,, \tag{2.53}$$

which determines how the renormalised coupling **runs with the energy** given its value at some reference scale μ_0. From Eq. (2.53) we see that the qualitative behavior of the RGE evolution of the renormalised coupling depends on the sign of the coefficient b_0, which depends on the specific theory:

- For $b_0 < 0$: $\alpha_{\rm ren}(\mu^2)$ *increases* with the energy μ^2.
- For $b_0 > 0$: $\alpha_{\rm ren}(\mu^2)$ *decreases* with the energy μ^2.

In the case of QED, we have that $b_0 < 0$ which leads to a renormalised electromagnetic coupling which becomes larger with the energy scale of the process under consideration.

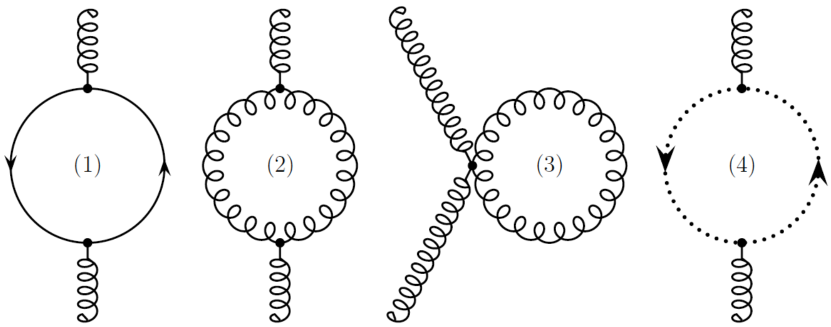

Figure 2.4. Feynman diagrams which contribute to the self-interactions of the gluon in QCD, and determine the leading order coefficient of the RGE equations for the running of $\alpha_{\rm ren}(\mu)$ in Eq. (2.52). The fourth diagram represents the contribution from *ghosts* arising in non-Abelian QFTs.

The running coupling in QCD. The leading order term for the QCD beta function is given by

$$b_0 = \frac{33 - 2n_f}{12\pi} > 0, , \tag{2.54}$$

where n_f is the number of active quarks in the theory ($n_f^{(\rm max)} = 6$). The fact that b_0 is positive arises from the gluon self-interactions, that is, from the non-abelian nature of QCD. To see this, note that the diagram with quarks in the loop leads to the n_f-proportional piece, diagram #1 in Fig. 2.4. Looking at Eq. (2.53), we see that $b_0 > 0$ implies that the theory becomes non-interacting in the ultra-violet, hence the name *asymptotically free*.

In the case of QED, the first term of the beta function is negative, due to the absence of photon self-interactions in the QED Lagrangian:

$$b_0^{\rm QED} = -\frac{4n_f}{12\pi} \tag{2.55}$$

As opposed to QCD, QED becomes strongly interacting (non-perturbative) in the ultraviolet, that is, at very small distances and large scales. The scale where QED stops being weakly coupled is determined by the position of the Landau pole in QED, which is

$$\Lambda = m_e \exp\left(-\frac{1}{2b_0\alpha_{\rm ren}(m_e)}\right) \sim 10^{90}\,\text{GeV} \gg M_{\rm pl} \tag{2.56}$$

much larger than any sensible scale, even the Plank scale where quantum gravity effects become important.

In summary, the renormalization group equations allows to determine the dependence of the running coupling on the scale, in terms of $\alpha_S(Q_{\rm ref})$ at some reference scale, which needs to be extracted from experimental data. In Fig. 2.5 we summarize recent experimental determinations of the QCD running coupling $\alpha_s(Q)$ for different scales, together with the four-loops QCD prediction. Asymptotic freedom can be seen by the fact that $\alpha_s(Q)$ decreases when Q is increased.

Remarkably, we still don't understand what are the dynamics that lead to confinement. The QCD coupling is large at low scales, but there perturbation theory breaks down. Confinement is an experimental observation (hadrons are always color singlets) but it is a purely non-perturbative phenomenon, thus non-perturbative methods are required, like lattice QCD. In Fig. 2.5 we show how in a lattice QCD calculation the quark-antiquark potential increases linearly with the with the separation between the quark and the antiquark, as expected in a confining theory, but still the relevant degrees of freedom are quarks and gluons, not hadrons. Nevertheless, the mechanism that leads to the realization of confinement remains still elusive.

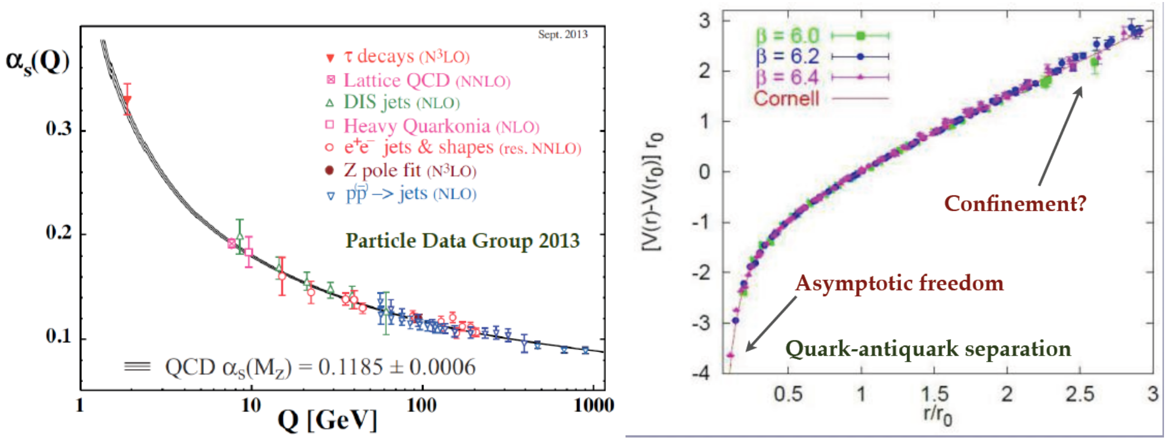

Figure 2.5. Left plot: summary of determinations of the QCD running coupling $\alpha_S(Q)$ for different scales, together with the four-loops QCD prediction. Asymptotic freedom can be seen by the fact that $\alpha_s(Q)$ decreases when Q is increased. Right plot: calculation of the potential between a quark-antiquark pair as a function of their separation in lattice QCD, showing the increase in their attraction as their separation grows.

Discussion #2.8

Could QCD exist as a purely gluonic theory, that is, with $n_f = 0$, or would such theory become nonphysical? Does a gluon-only version of QCD exhibit any **pathological behaviour**?