Effective Field Theory

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1 Lecture 1: Introduction

1.1 What is an EFT?

Some theorists perspective of physics: want to obtain a theory of everything!
Effective field theory (EFT) takes the opposite perspective: only describe the *relevant degrees of freedom.* Intuitively, we do not need to know about the top quark to describe the the proton, since its mass $m_t \sim 172$ GeV $\gg 1$ GeV $\sim m_p$. That doesn't mean that the top quark effects are negligible: in one of the exercises you will show that $m_p \propto m_t^{2/23}$. Rather, the effect of the top quark (high energy physics) has been *absorbed* in the proton mass (low energy parameter), that we determined experimentally. Similarly, we don't need to know about quantum gravity to describe collisions at the Large Hadron Collider (which doesn't mean that their effects are small).

EFTs take into account that there is interesting physics at a wide range of scales:

- the scale $\Lambda_{\text{QCD}} \sim 1$ GeV where the strong force becomes confining
- the electroweak symmetry breaking scale $v = 246$ GeV.
- some scale $\Lambda_{\text{NP}} \gtrsim 10^3$GeV of physics beyond the Standard Model
- quantum gravity effects become important important at Planck scale $M_{pl} \sim 10^{18}$ GeV
- wide range of particle masses: $m_e \sim 5{\cdot}10^{-4}$ GeV, $m_p \sim$ GeV, $m_t \sim 173$ GeV
- the typical energy E probed by some experiment

EFT is doing QFT with expansion parameter(s). This is referred to as *power counting*, and the expansion parameter is not the coupling but the ratio of scales. It relies on the *principle of scale separation* outlined above: to describe physics at energy E we don't need dynamics at energies $M \gg E$. Instead the effect of high physics at the scale M is absorbed in parameters of the effective field theory Lagrangian.

Example 1: $\mu \to e\bar{\nu}_e\nu_\mu$ is a weak process **(Draw)**. Since $m_\mu \ll m_W$, we can effectively describe this by a four-fermion interaction. In fact, this is what historically happened (Fermi theory). More precise measurements would probe corrections:

$$\begin{aligned}\bar{\nu}_{\mu,L}\gamma^\alpha\mu_L\frac{-\mathrm{i}}{p^2-m_W^2}\bar{e}_L\gamma_\alpha\nu_{e,L} &= \mathrm{i}\,\bar{\nu}_{\mu,L}\gamma^\alpha\mu_L\left[\frac{1}{m_W^2}+\frac{p^2}{m_W^4}+\dots\right]\bar{e}_L\gamma_\alpha\nu_{e,L} \\ &= \mathrm{i}\,\bar{\nu}_{\mu,L}\gamma^\alpha\mu_L\left[\frac{1}{m_W^2}-\frac{\Box}{m_W^4}+\dots\right]\bar{e}_L\gamma_\alpha\nu_{e,L} \qquad (1.1)\end{aligned}$$

Two approaches to constructing EFTs:

- Top-down: high energy theory is known but effective theory offers advantages: additional symmetries, easier to calculate, etc. E.g. Fermi's theory of weak interactions or Heavy Quark Effective Theory (HQET).
- Bottom-up: high energy theory is not known, construct operators all terms in the Lagrangian consistent with symmetry up to required accuracy in the power expansion. E.g. Standard Model Effective Theory (SMEFT) or Chiral Perturbation Theory (χPT).

We do not require EFTs to be renormalizable in the traditional sense of the word. Rather, the EFT is renormalizable up to the order in the power counting that one is working. **(draw example of two ϕ^6 interactions requiring a ϕ^8 counterterm)** It is worth emphasizing that EFTs are full-fledged QFTs and don't require a complete high energy (UV) theory for loop corrections, even though these loop integral involve large momenta that would seem to probe the UV physics.

If the UV theory is known, one can determine the coefficients of interactions in the EFT by *matching:* UV theory = sum over operators (interactions) in EFT × matching coefficients. In the case of high and low-energy degrees of freedom H and ℓ, one can make this explicit at the level of the path integral:

$$\int \mathcal{D}H\, \mathcal{D}\ell\, \exp(\mathrm{i}S_{\mathrm{UV}}[H,\ell]) = \int \mathcal{D}\ell\, \exp(\mathrm{i}S[\ell]) \tag{1.2}$$

We can *integrate out* H because it is too high in energy (or mass) to appear as external state in an amplitude. Here $S[\ell]$ is non-local at short-distances $x \lesssim 1/M$, but at the larger distance scale $1/m$ of ℓ it can be expanded in local operators using the operator product expansion (OPE), see eq. (1.1).

EFTs provide a systematically-improvable expansion and are therefore *not modelling* in the sense of e.g. models of the proton, which are intrinsically limited. EFTs *can* be used to describe nonperturbative phenomena, as we will see in the example of χPT.

1.2 Why use an EFT?

EFTs allow you to parametrize unknown physics in a *model-independent* way. Example is the Standard Model Effective Theory (SMEFT), which is used at the LHC to parametrize physics beyond the Standard Model in terms of the coefficients of new interactions.

EFTs simplify calculations by allowing you to focus on one scale at the time. Concretely you can set up an effective theory at each relevant scale and match between them.

EFTs allow you to resum logarithms of the ratios of scales by using the renormalization group: Perturbative corrections in multi-scale problems typically involve logarithms of the ratio of scales. For far-apart scales, these logarithms are large, spoiling the expansion in the coupling. Explicitly, $\alpha^n \ln^m(\Lambda_1/\Lambda_2)$ is formally suppressed in the coupling α but this can be counteracted if the scales $\Lambda_{1,2}$ are far apart. EFTs allow you to address this.

By performing an expansion, new symmetries become manifest. For example, in the limit of large quark mass, the leading interactions with the quark are independent of its spin (and also independent of the heavy flavor).

1.3 Examples of EFTs

We will now discuss several examples of EFTs, to discuss the main features: determining the degrees of freedom, the power counting and the construction of the Lagrangian. This is meant to set the stage with a more in-depth discussion in the next lectures.

Example 2: Top-down approach for a heavy scalar H and light scalar ℓ

$$\begin{aligned}\mathcal{L} &= \frac{1}{2}\partial_\mu \ell \partial^\mu \ell - \frac{1}{2}m^2\ell^2 + \frac{1}{2}\partial_\mu H \partial^\mu H - \frac{1}{2}M^2 H^2 - V(\ell, H)\,, \\ V(\ell, H) &= \frac{g_\ell}{4!}\ell^4 + \frac{g_h}{4!}H^4 + \frac{g_{\ell h}}{4}\ell^2 H^2 + \frac{\tilde{m}}{2}\ell^2 H + \frac{\tilde{g}_h M}{3!}H^3\,. \end{aligned} \tag{1.3}$$

We imposed a $\ell \to -\ell$ symmetry and will assume the following hierarchy of scales: $M \gg m, \tilde{m}$. Calculating the amplitude for $\ell\ell$ scattering at tree-level **(Draw diagrams)**

$$\begin{aligned} \mathrm{i}\mathcal{A}(p_1, p_2, p_3, p_4) &= g_\ell + \tilde{m}^2\Big[\frac{1}{(p_1+p_2)^2 - M^2} + \frac{1}{(p_1-p_3)^2 - M^2} \\ &\quad + \frac{1}{(p_1-p_4)^2 - M^2}\Big] \\ &= g_\ell - \frac{3\tilde{m}^2}{M^2} - \frac{\tilde{m}^2}{M^4}[(p_1+p_2)^2 + (p_1-p_3)^2 + (p_1-p_4)^2] + \mathcal{O}\left(\frac{E^2}{M^6}\right) \\ &= g_\ell - \frac{3\tilde{m}^2}{M^2} - \frac{4\tilde{m}^2 m^2}{M^4} + \mathcal{O}\left(\frac{E^2}{M^6}\right). \end{aligned} \tag{1.4}$$

The last line follows from momentum conservation and the on-shell condition $p_i^2 = m^2$ (the known result for Mandelstam variables: $s + t + u = 4m^2$). In the effective theory the same result would have been reached using

$$\left(g_\ell - \frac{3\tilde{m}^2}{M^2} - \frac{4\tilde{m}^2 m^2}{M^4}\right)\frac{1}{4!}\ell^4\,. \tag{1.5}$$

In fact, all other low energy observables are reproduced at tree-level up to $\mathcal{O}(1/M^4)$ if the following effective Lagrangian is used

$$\mathcal{L}_{\text{eff}} = \frac{1}{2}\partial_\mu \ell \partial^\mu \ell - \frac{1}{2}m^2\ell^2 - \left(g_\ell - \frac{3\tilde{m}^2}{M^2} - \frac{4\tilde{m}^2 m^2}{M^4}\right)\frac{1}{4!}\ell^4 - \frac{\tilde{m}^2}{M^4}\left(\frac{g_{\ell h}}{16} - \frac{g_\ell}{6}\right)\ell^6\,. \tag{1.6}$$

One may ask the question why no terms involving derivatives appeared in eq. (1.6), such as $(\Box\ell)\ell^3$ or $(\partial_\mu\ell)^2\ell^2$, but these can be eliminated by using *partial integration* and the (leading) *equation of motion* $\Box\ell = -m^2\ell + \ldots$

$$(\Box\ell)\ell^3 \to -m^2\ell^4\,, \qquad 0 = \partial^\mu[(\partial_\mu\ell)\ell^3] = 3(\partial_\mu\ell)^2\ell^2 + (\partial^2\ell)\ell^3\,. \tag{1.7}$$

Scalar EFTs will be investigated in detail over the next two lectures.

Example 3: Standard Model Effective Theory (SMEFT): In this case the degrees of freedom are the Standard Model fields and the power expansion is in E/Λ_{NP} where E is the typical energy scale of experiments and Λ_{NP} the scale of new physics. The Standard Model symmetries are imposed on the effective theory, but higher dimensional operators are now also allowed. Their contributions are suppressed by appropriate powers of E/Λ_{NP}. In this case the full theory is not known, but one can match to a specific UV completion of the Standard Model, if desired. Interestingly, there is a unique dimension 5 operator:

$$\mathcal{L}_5 = -\frac{C_W^{(5)}}{\Lambda} L_k^T C L_m H_l H_m \epsilon^{kl}\epsilon^{mn} + \text{h.c.} \tag{1.8}$$

where L is the left-handed lepton doublet, H is the Higgs doublet and C is the charge conjugation matrix.

Inserting the Higgs minimum, this leads to a Majorana mass term:

$$\mathcal{L}_5 = -\frac{C_W^{(5)} v^2}{\Lambda} \nu_L^T C \nu_L \,. \tag{1.9}$$

While in the original Standard Model neutrinos were massless, we now know this is not the case due to neutrino oscillations. It is interesting that the first extension beyond the Standard Model remedies this.

Example 4: Chiral Perturbation Theory (χPT): The QCD Lagrangian is given by

$$\mathcal{L}_q = \bar{q}\, \mathrm{i}\not{D} q - \bar{q} M q = \bar{q}_L\, \mathrm{i}\not{D} q_L + \bar{q}_R\, \mathrm{i}\not{D} q_R - \bar{q}_L M q_R - \bar{q}_R M^\dagger q_L \tag{1.10}$$

where $q = (u, d, s)$, $q_{R,L} = \frac{1}{2}(1 \pm \gamma_5) q$ and $\not{D} = \gamma^\mu(\partial_\mu + \mathrm{i} g A_\mu^a t^a)$. The three-lightest quarks are almost massless, for which this Lagrangian has an (approximate) global chiral symmetry: $q_L \to U_L q_L, q_R \to U_R q_R$ with $U_L \in U(3)_L$ and $U_R \in U(3)_R$. We can decompose $U(3)_R \times U(3)_L = U(1)_V \times U(1)_A \times SU(3)_V \times SU(3)_A$, where V (A) means $L = R$ ($L = R^{-1}$). $U(1)_V$ corresponds to (conserved) baryon number, while $U(1)_A$ is broken by quantum effects (the famous triangle anomaly). $SU(3)_V$ underpins the structure of the observed hadron spectrum, Gell-Mann's eightfold way, since

$$\begin{aligned} Q_V|p\rangle &= Q_V a_p^\dagger|0\rangle = ([Q_V, a_p^\dagger] + a_p^\dagger Q_V)|0\rangle = a_n^\dagger|0\rangle = |n\rangle \,, \\ H|n\rangle &= H Q_V|p\rangle = ([H, Q_V] + Q_V H)|p\rangle = Q_V m_p|p\rangle = m_p|n\rangle \,. \end{aligned} \tag{1.11}$$

Here Q_V is one of the charges corresponding to $SU(3)_V$.

$SU(3)_A$ turns out to be spontaneously broken. If not, we would have expected parity doubling, which is not observed. The key difference in eq. (1.11) for $SU(3)_A$ is that $Q_A|0\rangle \neq 0$, so $Q_A|p\rangle$ is no longer a single particle state. The eight (psuedo) Goldstone bosons can be identified with the lightest mesons $\pi^0, \pi^\pm, K^0, \bar{K}^0, K^\pm, \eta$, and are psuedoscalars in agreement with $Q_A|0\rangle \neq 0$. They are not exactly massless due to the quark masses, but much lighter than other bound states. One can write down an effective theory for these Goldstone bosons that e.g. relate meson masses to quark masses

$$m_\pi^2 = B(m_u + m_d)\,, \quad m_{K^\pm}^2 = B(m_u + m_s)\,, \quad m_{K^0}^2 = B(m_d + m_s)\,, \tag{1.12}$$

or can be used to calculate $\pi\pi$ scattering. In this case the power counting is in E/Λ_χ, where $\Lambda_\chi \sim 1$ GeV is the chiral symmetry breaking scale. Although the UV theory is known (QCD), the matching can't be performed because it is nonperturbative - already the degrees of freedom of the two theories are different!. However, the unknown coefficients in the Lagrangian, such as B in eq. (1.12), can be determined from experimental data.

Example 5: Effective theory for quantum gravity: At energies E sufficiently below the Planck scale $M_{\rm pl}$, one can write down an EFT for quantum gravity. The starting point is the Einstein-Hilbert action for general relativity, but now we allow for higher dimension operators

$$S_{\rm grav} = \int \mathrm{d}^4 x \sqrt{-g} \Big[\frac{1}{2} M_{pl}^2 R + c_1 R^2 + c_2 R_{\mu\nu} R^{\mu\nu} + c_3 R_{\mu\nu\lambda\rho} R^{\mu\nu\lambda\rho} + \frac{c_4}{M_{pl}^2} R^3 + \dots\Big]\,, \tag{1.13}$$

where the first term is the usual Einstein-Hilbert action. Under the assumption of *naturalness* $c_i = \mathcal{O}(1)$, quantum gravity effects are suppressed by $\sim E^2/M_{pl}^2$. Leading effects require c_1 through c_3, though this basis can still be further reduced by using equations of motion (no matter fields included):

$$R^{\mu\nu} = \frac{1}{2} R g^{\mu\nu} \quad \to \quad R_{\mu\nu} R^{\mu\nu} = \frac{1}{4} g_{\mu\nu} g^{\mu\nu} R^2 = R^2 \,. \tag{1.14}$$

One can calculate the gravitational potential of two masses:

$$V(r) = -\frac{GMm}{r}\Big[1 + 3\frac{G(M+m)}{rc^2} + \frac{41}{10\pi}\frac{G\hbar}{r^2c^3} + \dots\Big]\,, \tag{1.15}$$

where (for once) the appropriate factors of c and $\hbar$ are included to highlight the relativistic and quantum origin of the second and third term. This is valid if $r \gg 1/M_{\rm pl}$. Interestingly, the leading quantum correction to this potential is independent of c_i.

Example 6: Effective theory for condensed matter: EFT is also used by condensed matter physicists who use field theory in their work. A fun example is the question of whether sound waves (or phonons) can carry mass. The answer is normally a no as energy and momentum can propagate but mass oscillates netting zero $\langle\phi\rangle = 0$. Standing waves in a cubical cavity oscillate over time as well but don't transfer anything over spacetime without adding some probe. However, this is the case for linear theories and like how we can't rid of the non-zero VEV in the Higgs mechanism due to it being a non-linear theory one can find mass being transferred in non-linear wave theories. The authors Nicolis and Penco found the interaction of phonons and rotons with gravity inside of a superfluid in the limit of absolute zero temperature to leave the quasi-particles with an "effective gravitational mass" transfer.
So, how heavy are these tunes? As is usual, it depends on the details of the medium (what material) and its condition (temperature). We should also be careful to remember that in a material the usual notion of a Poincaré invariant mass complicates due to us not being able to casually boost around anymore. For this we'll be interested in the non-relativistic limit where mass is conserved and we can think in Newtonian and Galilean terms. The mass M being the fraction of mass of the total mass of the field that propagates, the quantity being zero if none of the mass in the field propagates.
The action of interest, superfluid helium-4 in the $E(\mathbf{p}) \to 0$ limit, is given by

$$S = \int \mathrm{d}t\, L = \int \mathrm{d}t\, \left(\mathbf{p}\cdot\partial_t \mathbf{x} - H(|\mathbf{p}|)\right) \approx \int \mathrm{d}t\, p\left(|\partial_t \mathbf{x}| - c_s\right)\,, \tag{1.16}$$

where the spectrum consists of phonons respecting $E \simeq c_s p \approx (ma)^{-1} p$ ($\omega = c_s k$ wave) and where we assume the local minima holding rotons is far enough away for us to ignore for simplicity. Here a is the atomic distance and m the mass of helium-4. We want to introduce a "bulk perturbation" which consists of some some long wavelength modes spanning a large part of the material rather than a point. We introduce the bulk π modes $\mathbf{u} = -c^2\nabla\pi$, the lowest order invariant operator (velocity potential ϕ) $X = -c^2\partial^\mu\phi\partial_\mu\phi$ and a Newtonian gravitational potential Φ resulting in

$$S \simeq \int \mathrm{d}t\, p\left(|\partial_t \mathbf{x} - \mathbf{u}| - c_s(\sqrt{X})\right) \text{ with } X \to \mu^2\left(\left(1 - \frac{2\Phi}{c^2}\right)(1+\partial_t\pi)^2 - c^2(\nabla\pi)^2\right)\,, \tag{1.17}$$

with μ the chemical potential. The non-relativistic EFT whose EOM shows it's just a non-linear wave

$$S = \int \mathrm{d}^4x F(X) = \frac{\rho_m}{2c_s^2}\int d^4x\left((\partial_0\phi)^2 - c_s^2(\nabla\phi)^2 + \partial_0\phi(\nabla\phi)^2 - \frac{1-2c_s c_s'}{3c_s^2}\left(\partial_0\phi\right)^3\right) \tag{1.18}$$

$$\text{where } c_s^2 = \frac{F'}{F''} \text{ with } X = -\partial_t\phi - \frac{1}{2}\left(\nabla\phi\right)^2\,, \tag{1.19}$$

Where π and Φ don't need to be explicitly referenced to anymore. What is M, we know that $\rho = \partial_{\partial_0 \phi} F$ which implies

$$M = \langle \delta\rho \rangle = -\frac{\rho_m}{c_s^2} \langle \partial_0 \phi + \frac{1}{2} (\nabla\phi)^2 - \frac{1 - 2c_s c_s'}{2c_s^2} (\partial_0 \phi)^2 \rangle = -\frac{c_s'}{c_s} E \simeq -\frac{\mathrm{d} \log c_s}{\mathrm{d} \log \rho_m} \frac{E}{c_s^2}\,, \tag{1.20}$$

giving us our sound wave mass. The jump from the original theory to the EFT was quite large but nevertheless we obtained our main goal. We could have also expanded $\sqrt{X}$ and found that at leading order the gravitational potential shifts the speed of sound resulting in a non-zero M. This also happens in light-heavy scalar l, H theory where finding the matrix element using an EFT where $m_l < m_H$ produces the same result as calculating that same matrix element in the full theory but then taking the $m_l < m_H$ limit.

There are many more EFTs one can consider: nonrelativistic QCD relevant for describing certain bound states, heavy quark effective theory, effective theories of gravity for non-relativistic extended objects...

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2 Lecture 2: Scalar EFT part I

Learning Goals

- Revise basic concepts of QFT in case of scalar field theories.
- Applying the EFT paradigm to a toy model composed by two scalar fields.
- Derive matching relations at tree level in scalar EFTs.

In this lecture and the following, we illustrate the main concepts of the Effective Field Theory paradigm in the case of relatively simple scalar QFTs, in preparation for the discussion of more realistic EFTs taking place in subsequent lectures. First of all we review basic concepts in QFT, in particular the ideas of regularisation and renormalisation applied to scalar QFTs. Then we introduce a toy model for a scalar EFT, composed by two scalar fields, one heavy and the other light. Finally we demonstrate how to match calculations carried out in the EFT and in the UV theory in order to determine the values of the Wilson coefficients of the EFT.

2.1 Regularisation and renormalisation in QFTs

Let us begin with a brief recap of the basic ideas of regularisation and renormalisation in QFTs, focusing on the specific case of scalar QFTs. In quantum field theory, renormalisation is the method that allows one to make sense of a theory which contains ultraviolet (UV) divergences. Such UV divergences are a generic feature of most QFTs.

Let us consider for example the $\lambda\phi^3$ theory, a theory containing a single real scalar field with an interaction term of the form

$$\mathcal{L}_{\phi^3} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{3!}\phi^3 \,, \tag{2.1}$$

and where the first term is the kinetic term of the theory. The self-interaction (two-point function) one-loop diagram, see Fig. 2.1 (left), results in a loop integral of the form

$$\propto \lambda^2 \int \frac{d^4q}{(2\pi)^4}\frac{1}{(q^2-m^2)((q+k)^2-m^2)} \,, \tag{2.2}$$

with k^μ being the four-momentum flowing into the loop.

If we focus on the UV integration region where $q^\mu \to \infty$ then one sees that the loop integral Eq. (2.2) simplifies to

$$\sim \lambda^2 \int \frac{d^4q}{(2\pi)^4}\frac{1}{q^4} \sim \int_c^\infty \frac{q^3dq}{q^4}\int d\Omega_4 \,, \tag{2.3}$$

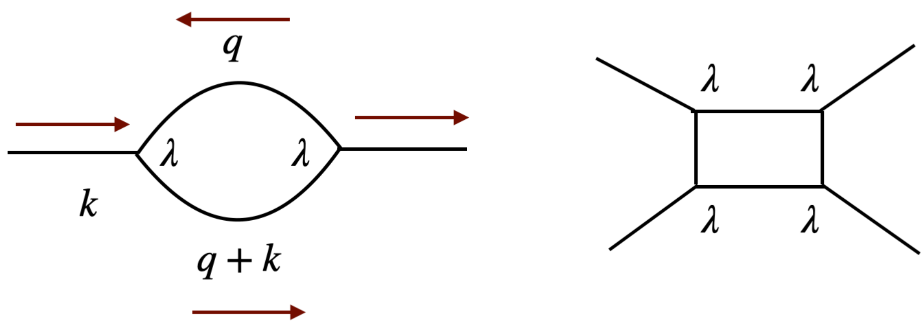

Figure 2.1: Loop corrections arising in the $\lambda\phi^3$ theory for the two-point (left) and four-point (right panel) functions.

where in the last step we have transformed to n-dimensional spherical coordinates in $\mathbb{R}^4$, and c is some real-valued constant. The integral over the radial coordinate is logarithmically divergent, and hence the loop integral Eq. (2.2) is ill-defined. Other loop integrals in the same theory may instead be convergent, for example the one-loop correction to the four-point scattering amplitude Fig. 2.1 (right) scales in the UV as

$$\sim \lambda^4 \int \frac{d^4q}{(2\pi)^4}\frac{1}{(q^2)^4} \sim \int_c^\infty \frac{dq}{q^5}\int d\Omega_4\,, \tag{2.4}$$

which converges, ridding us of the UV divergence.

Regularization. The first step to make sense of UV divergences is to regularise the divergent integrals. For instance, one can regularise using a cutoff Λ,

$$\int_c^\infty \frac{q^3dq}{q^4} \to \int_c^\Lambda \frac{q^3dq}{q^4}\,, \tag{2.5}$$

an approach which is however not Lorentz-invariant. Here we will work in the frequently adopted dimensional regularisation prescription

$$\int d^4q \to \int d^nq\,, \qquad n = 4-2\epsilon\,, \tag{2.6}$$

with n being the number of space-time dimensions and where in the limit of $\epsilon \to 0$ we recover the physical $n = 4$ case. It is now possible to formally compute the outcome of loop integrals such as Eq. (2.2) by using the following result

$$I(n,\alpha) = \int \frac{d^nq}{(2\pi)^n}\frac{1}{(q^2-m^2)^\alpha} = \frac{i\pi^{n/2}}{(2\pi)^n}\frac{\Gamma(\alpha-n/2)}{\Gamma(\alpha)}(-1)^\alpha\left(m^2\right)^{\frac{n}{2}-\alpha}\,, \tag{2.7}$$

in terms of the Gamma function $\Gamma(x)$. Note that the divergence has not gone away: in the case that corresponds to Eq. (2.2), namely $\alpha = 2$ and $n = 4$, one still has $\Gamma(0)$ which is divergent. One advantage of expressing divergent loop integrals in terms of Gamma functions is that they can be continued analytically, for example using the formula

$$z\Gamma(z) = \Gamma(z+1)\,, \quad z \in \mathbb{C}\,. \tag{2.8}$$

In the case of interest, $\alpha = 2$ and $n = 4-\epsilon$, we can expand $I(n,\alpha)$ around the divergence at $\epsilon \to 0$ by using the following relation:

$$I(n,\alpha) = \frac{i\pi^{n/2}}{(2\pi)^n}\Gamma\left(2-\frac{n}{2}\right)\left(m^2\right)^{n/2-2} = \frac{i}{16\pi^2}\left(4\pi\right)^{\epsilon}\Gamma(\epsilon)\left(m^2\right)^{-\epsilon} \tag{2.9}$$

$$I(n,\alpha) \simeq \frac{i}{16\pi^2}\left(\frac{1}{\epsilon} - \gamma_E + \ln 4\pi - \ln m^2 + \mathcal{O}\left(\epsilon\right)\right) , \tag{2.10}$$

where we have used the usual Gamma function expansion

$$\Gamma(\epsilon) = \frac{1}{\epsilon}\Gamma(1+\epsilon) \simeq \frac{1}{\epsilon} - \gamma_E + \mathcal{O}\left(\epsilon\right) . \tag{2.11}$$

As you have seen in previous courses, you can bring loop integrals into the form of $I(n,\alpha)$ by means of the Feynman parameterization technique, namely

$$\frac{1}{AB} = \int_0^1 dx \frac{1}{\left[xA + (1-x)B\right]^2} \tag{2.12}$$

$$\frac{1}{ABC} = 2\int_0^1 dx \int_0^{1-x} dy \frac{1}{\left[xA + yB + (1-x-y)C\right]^3} , \tag{2.13}$$

and likewise.

A special and important case of loop integrals arising in QFTs is the case of integrals where

$$\int d^n q \frac{1}{(q^2)^{\alpha}} = 0 \tag{2.14}$$

for any value of α. These integrals are scale-less, and are always set to zero in dimensional regularisation. These scale-less integrals, without any external mass scale involved, are also known as massless tadpoles.

Renormalization. The regularisation procedure enables clearly separating the divergence term from the finite part of the calculation, in the case at hand the $1/\epsilon$ pole in Eq. (2.10). But the integral remains divergent nevertheless.

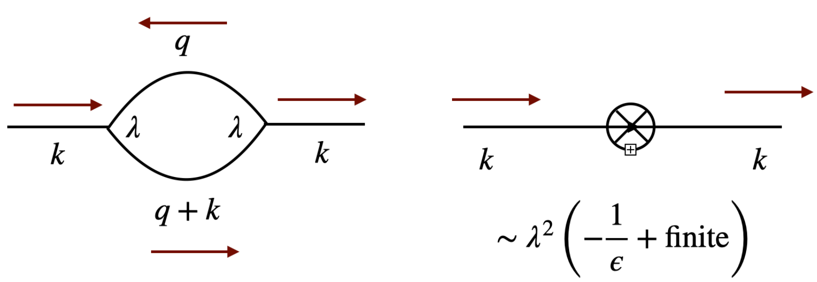

Figure 2.2: In the $\lambda\phi^3$ scalar theory, the UV divergence associated to the the two-point function (self-energy) is cancelled by the addition of the appropriate counterterm (right panel).

To renormalize such divergences, one adds counterterms to the Lagrangian such that the infinities cancel e.g. the one-loop divergence in the propagator of $\lambda\phi^3$ theory is cancelled by the contribution of the counterterm, chosen to cancel the loop divergence, as indicated in Fig. 2.2,

$$\sim \lambda^2 \left(-\frac{1}{\epsilon} + \text{finite} \right) . \tag{2.15}$$

Therefore the renormalized QFT consists of the original Lagrangian plus all the counterterms required to cancel out the UV divergences of the theory. Two important comments in this respect:

- The physical (observable) coupling constants and masses remain finite in the renormalised QFT.
- A scale μ is introduced to help keeping dimensional counting consistent. This means that the following dimensional analysis holds:

$$\left[\int d^4 q \right] = 4 \, , \quad \left[\int d^n q \right] = n \, , \tag{2.16}$$

$$\mu^{2\epsilon} \left[\int d^n q \right] = 4 \, \to \, \ln \frac{m^2}{\mu^2} \text{ terms} \, . \tag{2.17}$$

In general, renormalised parameters will depend on the arbitrary scale μ introduced by the renormalisation procedure: $\lambda(\mu)$, $m(\mu)$, etc. The higher the order in perturbation theory at which a given observable is computed, the less important the dependence with the arbitrary scale μ is expected to be.

In this lecture we will introduce a toy scalar EFT with two scalar fields, a light one and a heavy one. In order to prepare ourselves for this discussion, it is worth first to revisit the main aspects of the well-known $\lambda\phi^4$ scalar theory.

2.2 The scalar $\lambda\phi^4$ theory revisited

Let us consider now another scalar QFT, namely $\lambda\phi^4$, which you are likely to have seen already. This theory is renormalisable, meaning that all its UV divergences can be cancelled by a finite number of counterterms. The starting point is the “bare” Lagrangian

$$\mathcal{L} = \frac{1}{2} \left(\partial_\mu \phi^{(0)} \right)^2 - \frac{1}{2} m_0^2 \left(\phi^{(0)} \right)^2 - \frac{1}{4!} \lambda_0 \left(\phi^{(0)} \right)^4 \, , \tag{2.18}$$

which upon the addition of the suitable counterterms becomes the renormalised Lagrangian

$$\mathcal{L} = \frac{1}{2} \left(\partial_\mu \phi \right)^2 - \frac{1}{2} m^2 \phi^2 - \frac{1}{4!} \lambda \phi^4 + \text{counterterms} \, , \tag{2.19}$$

expressed in terms of the physical (observable) masses and couplings. The relation between the bare and renormalised quantities is the following

$$\phi^{(0)} = \sqrt{Z_\phi} \phi \, , \quad m_0 = Z_m m \, , \quad \lambda_0 = \mu^{2\epsilon} Z_\lambda \phi^4 \, , \tag{2.20}$$

and the Feynman rules of the renormalised $\lambda\phi^4$ theory are shown in Fig. 2.3. The renormalisation factors Z_i can be computed order by order in perturbation and are equal to one if one restricts ourselves to tree-level calculations.

The Feynman rules of the renormalised theory summarised in Fig. 2.3 can be directly read from the renormalised Lagrangian, starting from the bare Lagrangian and then using the relations in Eq. (2.20) in

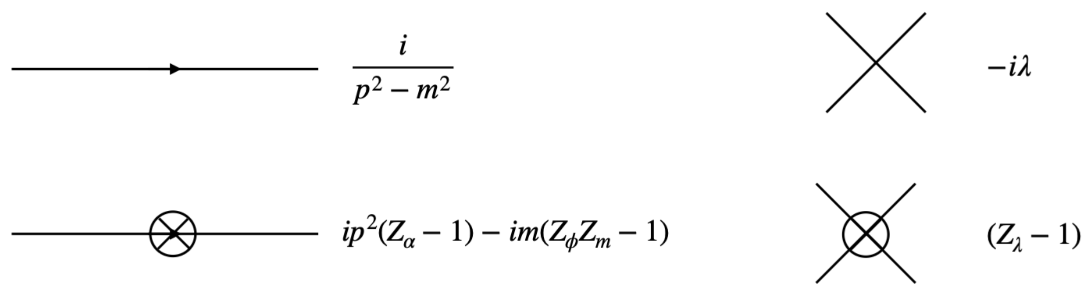

Figure 2.3: Feynman rules of the renormalised $\lambda\phi^4$ theory.

order to obtain

$$\begin{aligned} \mathcal{L} &= \frac{1}{2}\left(\partial_\mu \phi\right)^2 - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\mu^{2\epsilon}\phi^4 \\ &+ (Z_\phi - 1)\frac{1}{2}\left(\partial_\mu \phi\right)^2 + \left(Z_\phi Z_m - 1\right)\frac{1}{2}m^2\phi^2 + \left(Z_\lambda Z_\phi^2 - 1\right)\mu^{2\epsilon}\frac{\lambda}{4!}\phi^4 \,, \end{aligned} \tag{2.21}$$

where the bottom line corresponds to the contribution from the counterterms, and vanishes for tree-level calculations. The functions Z_i are typically of the form $Z_i = 1 + c_i \lambda^2 \frac{1}{\epsilon} + \mathcal{O}\left(\lambda^4\right)$ where the coefficients c_i need to be determined from specific calculations.

For instance, let us consider the one-loop corrections to the propagator in order to determine the Z_m and Z_ϕ functions at first order in the perturbative expansion, see Fig. 2.4. The amplitude includes the loop

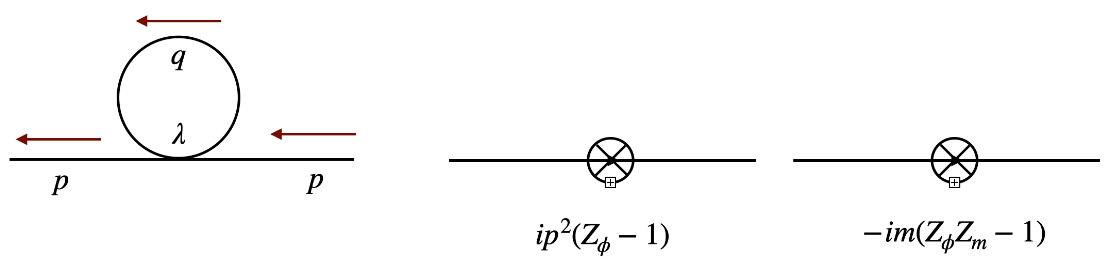

Figure 2.4: One loop corrections to the self-energy in $\lambda\phi^4$ theory, including the two counterterms.

diagram as well as the corresponding counterterms. The loop diagram is given by

$$-\frac{i\lambda}{2}\frac{\mu^{2\epsilon}}{(2\pi)^n}\int d^n q \frac{i}{(q^2 - m^2)} = -\frac{i\lambda}{2}\frac{i}{16\pi^2}(4\pi)^\epsilon \Gamma(-1+\epsilon)(m^2)^{1-\epsilon}\mu^{2\epsilon} \,. \tag{2.22}$$

Expanding the Gamma function around the singular region and adding the contribution from the counterterms results in the total one-loop correction to the propagator in the renormalised theory:

$$= i\lambda \frac{m^2}{32\pi^2}\left(\frac{1}{\epsilon} - \gamma_E + \ln 4\pi + 1 - \ln\frac{m^2}{\mu^2}\right) + ip^2(Z_\phi - 1) - i(Z_\phi Z_m - 1)m^2 \,, \tag{2.23}$$

where note the appeareance of the logarithm in the renormalisation scale μ. It is easy to see that the UV divergences cancel provided that we choose the following form for the counterterms

$$Z_\phi(\lambda) = 1 + \mathcal{O}(\lambda^2)\,, \tag{2.24}$$

$$Z_m(\lambda) = 1 + \frac{\lambda}{32\pi^2}\left(\frac{1}{\epsilon} - \gamma_E + \ln 4\pi\right) + \mathcal{O}(\lambda^2)\,. \tag{2.25}$$

A related calculation can be used to determine the coupling counterterm at one loop, which is given by

$$Z_\lambda = 1 - \frac{6\lambda^2}{32\pi^2}\left(\frac{1}{\epsilon} - \gamma_E + \ln 4\pi\right)\,, \tag{2.26}$$

and can be obtained by computing the one-loop corrections to the four-point scattering amplitude, $\phi + \phi \to \phi + \phi$ by imposing that UV divergences cancel out.

2.3 A two-particle scalar EFT

Following this recap of the basic features of renormalization and regularization in QFTs, and its application to scalar QFTs, let us now illustrate how the EFT paradigm applies to scalar field theories. We will consider the case of a two-particle scalar theory composed by a light field ϕ_ℓ with mass m_ℓ and a heavy field Φ with mass M.

The Lagrangian of this theory is given by

$$\mathcal{L}_{\rm UV} = \frac{1}{2}\left(\partial_\mu \phi_\ell\right)^2 - \frac{1}{2}m_\ell^2\phi_\ell^2 + \frac{1}{2}\left(\partial_\mu \Phi\right)^2 - \frac{1}{2}M^2\Phi^2 - V(\phi, \Phi)\,, \tag{2.27}$$

and where the scalar potential $V(\phi_\ell, \Phi)$ is given by

$$V(\phi_\ell, \Phi) = \frac{g_\phi}{4!}\phi_\ell^4 + \frac{g_\Phi}{4!}\Phi^4 + \frac{g_{\phi\Phi}}{4}\phi_\ell^2\Phi^2 + \frac{\tilde{g}_\phi}{2}M\phi_\ell^2\Phi + \frac{\tilde{g}_\Phi}{2}M\Phi^3\,, \tag{2.28}$$

where we have imposed that the potential is invariant under the $\phi_\ell \to -\phi_\ell$ transformation (this is a choice that specifies the global symmetries of the theory). The Lagrangian of Eq. (2.27) defines our UV theory.

We would like to construct an EFT that derives from this UV theory and which is valid for energy scales such that the heavy scalar particle Φ with mass M can be integrated out, that is, we are interested in the kinematic regime such that

$$\frac{p}{M}, \frac{m_\ell}{M} \ll 1\,. \tag{2.29}$$

Intuitively, it should be clear that in this regime the heavy scalar field Φ should not matter much. However, this does not mean that its effects can be altogether discarded: they will still be present at low energies via effective interactions between the light degrees of freedom of the theory.

We want therefore to determine the mapping such that

$$\mathcal{L}_{\rm UV} \to \mathcal{L}_{\rm EFT}\,, \tag{2.30}$$

where in the low-energy Lagrangian $\mathcal{L}_{\rm EFT}$ the only dynamical degree of freedom is the light scalar field ϕ_ℓ. From dimensional considerations, and exploiting the fact that the UV theory contains a $\phi_\ell \to -\phi_\ell$ symmetry, the most general form of the low-energy Lagrangian $\mathcal{L}_{\rm EFT}$ will be

$$\mathcal{L}_{\rm EFT} = \frac{1}{2}\left(\partial_\mu \phi\right)^2 - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4 + \frac{c_6}{\Lambda^2}\phi^6 + \cdots \tag{2.31}$$

where

- Λ has units of energy and is required by the power-counting in $\delta \equiv p/\Lambda$, where p indicate either external momenta or light masses.
- The physics interpretation of Λ is that of a cutoff scale: at energy scales above Λ the EFT expansion breaks down because the heavy fields become also active degrees of freedom.
- c_6 here is known as a "Wilson coefficient" and plays the role of coupling constant of the low-energy EFT. Its value can be fixed by matching to the UV theory including the heavy degree of freedom, as we show next.
- Note that we denote the light scalar field as ϕ_ℓ in the UV theory and ϕ in the EFT, and likewise for its mass. The reason for this is that the two fields are the same at tree-level, but do not exactly coincide once we account for loop corrections so it is important to keep the notation separated.

Inserting the operator $c_6\phi^6/\Lambda^2$ in the tree level amplitude leads to an effect

$$\sim \delta^n\,, \quad n = d-4\,, \quad (\text{where } d=6)\,, \tag{2.32}$$

thus the dimension-6 operator scales as $1/\Lambda^2$, the dimension-8 operator scales as $1/\Lambda^4$, and so on, in $d = 4$ space-time dimensions.

2.4 Tree level matching in scalar EFTs

Before going on with the more formal discussion of scalar EFTs, let us try to determine the Wilson coefficient c_6/Λ^2 at tree level. For this, we use the matching procedure which will appear repeatedly through this course.

The recipe for tree-level (on-shell) matching between an EFT and its UV counterpart is the following:

- Compute the same scattering amplitudes using the EFT Lagrangian, $\mathcal{L}_{\text{EFT}}$, and its UV counterpart, $\mathcal{L}_{\text{UV}}$.
- The two amplitudes describe the same underlying process in different energy regimes: by comparing them taking the appropriate kinematic limit, we can infer the low-energy Wilson coefficients, in this case c_6, in terms of the masses and couplings of the heavy degrees of freedom in the UV theory.
- Note that at tree level we can ignore the counterterms of the renormalized theory and also that $\phi_\ell = \phi$ and likewise also for the associated masses.

Let's apply this procedure to the scalar field theory under consideration. First we match the tree-level physical mass for the light field

$$\text{UV } (\phi_\ell) \quad \frac{i}{p^2 - m_\ell^2}\,, \qquad \text{EFT } (\phi) \quad \frac{i}{p^2 - m^2}\,, \tag{2.33}$$

which as mentioned above results in the expected relation $m_\ell = m$. Note that however this simple relation between the light field mass in the UV and in the EFT will not be valid once we account for loop corrections.

Second, we match the four point amplitude in order to determine the tree-level quartic coupling. The relevant Feynman diagrams in the EFT and in the UV theory are shown in Fig. 2.5, and lead to the following tree-level contributions to the four-point scattering amplitude of light fields:

$$\mathcal{A}_{\text{UV}}(\phi + \phi \to \phi + \phi) = -ig_\phi + (-i\tilde{g}_\phi M)^2 \left(\frac{i}{s - M^2} + \frac{i}{t - M^2} + \frac{i}{u - M^2} \right)\,, \tag{2.34}$$

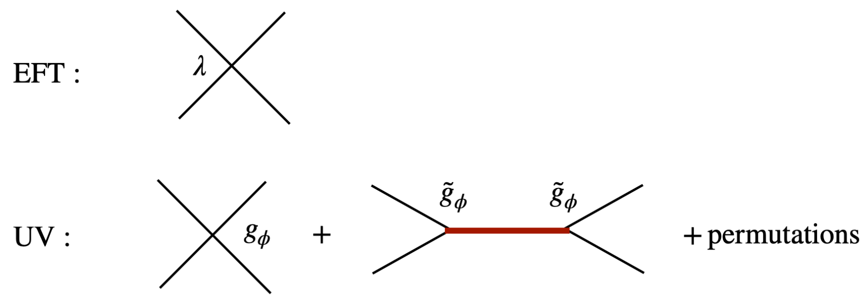

Figure 2.5: Feynmann diagrams contributing to the four-point amplitude $\phi\phi \to \phi\phi$ in the EFT and in the UV theory, where the thick lines indicate heavy quark lines in the UV theory.

$$\mathcal{A}_{\rm EFT}(\phi+\phi\to\phi+\phi) = -i\lambda\,. \tag{2.35}$$

Furthermore, for this scattering amplitude the can impose the following relation between the three Mandelstam variables: $s+t+u=4m^2$.

Now we recall that the EFT is an expansion in terms of the small parameter $\delta = m/M$, so we can expand the heavy line propagators appearing in the UV amplitude in powers of δ in order to obtain

$$\frac{i}{s-M^2} = \frac{-i}{M^2}\left(1+\frac{s}{M^2}+\mathcal{O}(\delta^4)\right)\,, \tag{2.36}$$

and likewise for the other two propagators, such that the UV four-point amplitude reads

$$\begin{aligned} \mathcal{A}_{\rm UV} &= -ig_\phi + i\tilde{g}_\phi^2\left(3+\frac{s+t+u}{M^2}\right)+\mathcal{O}(\delta^4) & (2.37)\\ &= -ig_\phi + i\tilde{g}_\phi^2\left(3+\frac{4m^2}{M^2}\right)+\mathcal{O}(\delta^4)\,. & (2.38) \end{aligned}$$

Therefore, we can write down the coupling constant of the ϕ^4 interaction at low energies (EFT) in terms of the couplings and masses of the UV Lagrangian, namely

$$\lambda = g_\phi - \tilde{g}_\phi^2\left(3+\frac{4m^2}{M^2}\right)+\mathcal{O}(\delta^4)\,. \tag{2.39}$$

While this matching procedure fixes the coupling λ associated to the 4-point interaction in the EFT, it leaves the six-point interaction unconstrained. In order to obtain a result for c_6/Λ^2 we need to consider higher-order scattering amplitudes, in particular the 6-point function. In Fig. 2.6 we show representative tree-level Feynman diagrams contributing to the six-point amplitude in the EFT (top) and in the UV theory (bottom), where heavy scalar propagators are indicated with a thick red line.

Some considerations concerning Fig. 2.6 can be raised at this point:

- The number of Feynman diagrams contributing to the six-point amplitude in the UV theory is much larger than for the same amplitude in the EFT.
- Diagrams containing several heavy scalar lines probably won't contribute to the Wilson coefficient c_6. In any case we should be careful since the heavy scalar mass M^2 may appear in the numerators.

In order to determine the coefficient c_6, we must hence solve the equation relating the two six-point

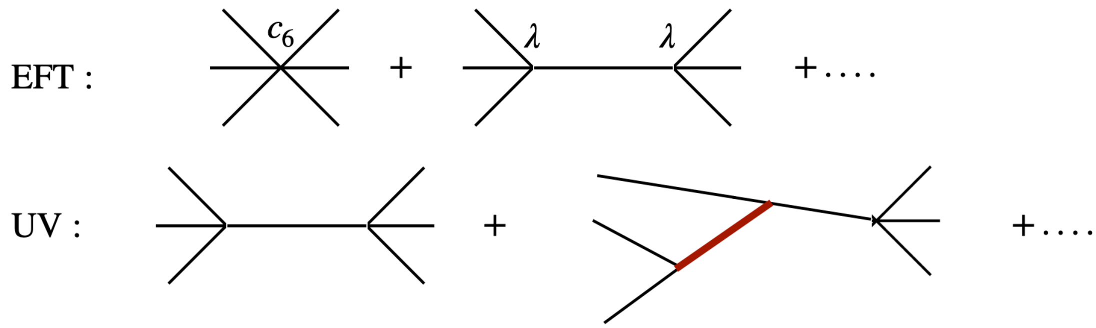

Figure 2.6: Representative tree-level Feynman diagrams contributing to the six-point amplitude in the EFT (top) and in the UV theory (bottom). Heavy scalar propagators are indicated with a thick red line.

amplitudes in the EFT and in the UV theory:

$$\mathcal{A}_{\mathrm{UV}}\left(\phi\phi\phi \to \phi\phi\phi\right) = \mathcal{A}_{\mathrm{EFT}}\left(\phi\phi\phi \to \phi\phi\phi\right) , \tag{2.40}$$

where the equation for c_6 needs to be solved order by order in the EFT expansion parameter δ. Let us see what do we get in this case. The second diagram in the top part of Fig. 2.6 (EFT amplitude) can be evaluated to yield

$$(-i\lambda)\frac{i}{\left(p_1+p_2+p_3\right)^2-m^2}(-i\lambda)\,, \tag{2.41}$$

with the ϕ^4 coupling in the EFT given by Eq. (2.39). We can carry out the analog calculation in the UV theory in order to obtain

$$\left(-ig_\phi + i\tilde{g}_\phi^2\left(3+\frac{3m^2+\left(p_1+p_2+p_3\right)^2}{M^2}\right)\right)\frac{i}{\left(p_1+p_2+p_3\right)^2-m^2}\left(-ig_\phi + i\tilde{g}_\phi^2\left(3+\frac{3m^2+\left(p_1+p_2+p_3\right)^2}{M^2}\right)\right)$$

where one "m" has been replaced by $p_{123}^2 \equiv \left(p_1+p_2+p_3\right)^2$. Therefore the EFT and UV diagrams don't quite cancel out in the equation once we substitute λ by the matched result of Eq. (2.39). The leftover difference can be checked to be

$$\frac{1}{M^2}\left(2i\tilde{g}_\phi^2\left(g_\phi - 3\tilde{g}_\phi^2\right)\right) + \mathcal{O}\left(\delta^2\right) , \tag{2.42}$$

which results in a contribution to the c_6 Wilson coefficient of the following form:

$$\frac{c_6}{\Lambda^2} = 20\tilde{g}_\phi^2\frac{\left(g_\phi - 3\tilde{g}_\phi^2\right)}{M^2} + \mathcal{O}\left(\tilde{g}_\Phi, g_{\phi\Phi}\right) . \tag{2.43}$$

This is not the full result for c_6, but illustrates how the matching procedure can provide a link between low-energy couplings and the couplings and masses of the heavy degrees of freedom in the UV theory.

In summary, we have shown in this section how to match at tree level the UV physics with the parameters of the EFT in the case of a two-particle scalar QFT. In the subsequent lecture we will show how this result can be extended to the case of one-loop corrections. Before that we will demonstrate the general procedure to set up an EFT from the bottom up, instead than the top-down construction that has been used in this lecture.

Effective Field Theory

Current version: **June 13, 2024**

3 Lecture 3: Scalar EFT part II

Learning Goals

- Constructing scalar EFTs from the bottom up.
- Remove redundancies present in an EFT operator basis.
- Derive matching relations in EFTs in the presence of loop corrections.
- Implement the decoupling method and the method of regions to streamline EFT calculations.

In the previous lecture, we have illustrated the basic methods underlying EFTs in the case of scalar field theories, specifically with a two-scalar model with a light and a heavy field. We have shown how at low energies the heavy degree of freedom can be integrated out, and how we can related the parameters (massess and couplings) of the UV theory to the Wilson coefficients arising in the subsequent EFT.

In this lecture we continue our study of the Effective Field Theory paradigm in the case of relatively simple scalar QFTs. In particular, after having shown how to match EFTs at the tree level, here we will demonstrate how to extend the procedure to the case of one-loop calculations. This discussion will also involve understanding how we can remove possible redundancies present in an operator basis, as well as explore methods that facilitate calculations in EFTs. Our discussion will highlight the potentialities and challenges and constructing EFTs from the bottom up, as required in the case for which the corresponding UV theory is not known.

3.1 Construction of an EFT from the bottom up

As we discussed in the previous lecture, in order to construct an EFT from the bottom up we need to:

- Adopt a given power counting for the EFT expansion.
- Impose the desired local and global symmetries that the EFT must satisfy, reflecting the symmetries that are expected for the UV theory.
- Construct all possible operators, order by order in the EFT expansion, by combining the degrees of freedom such that the required symmetries are satisfied.

In the scalar EFT example that we considered in the previous lecture, we have a single light degree of freedom, the light scalar field ϕ with mass m, since the heavy scalar field (Φ, with mass M) had been integrated out[1]. We assume that this EFT description is valid up to $E \simeq \Lambda$, with Λ being the mass scale at which the EFT expansion breaks down, hence $\Lambda \sim M$ (but in general we would not know this information). Hence the expansion parameter for our EFT will be $\delta \sim m/\Lambda, p/\Lambda$ with p indicating the momenta flowing through the EFT amplitudes.

Therefore, starting from the bottom-up and making minimal assumptions about the physics of the UV, the Lagrangian of the scalar EFT that we need to construct is composed by:

- All possible operators involving only the scalar field ϕ, including possible derivative couplings.
- These operators are ordered in powers of the expansion parameter δ.
- These operators satisfy the 'parity conservation" symmetry $\phi \to -\phi$ that is inherited from the physics of the UV (where we had a $\phi_\ell \to -\phi_\ell$ symmetry).

Taking into account these considerations, we write down all possible operators $\mathcal{O}_i^{(d)}$ composed by the degrees of freedom of the EFT (in our case, the light scalar field ϕ) which satisfy a specific set of symmetries (in this case, parity conservation):

$$\mathcal{L}_{\rm EFT} = \frac{1}{2}\left(\partial_\mu \phi\right)^2 - \frac{1}{2}m_\phi^2\phi^2 - \frac{\lambda}{4!}\phi^4 + \sum_{d\geq 6}\sum_{i}^{n_d}\frac{c_i^{(d)}}{\Lambda^{d-4}}\mathcal{O}_i^{(d)}\,, \tag{3.1}$$

where the sum over d includes only even terms to ensure that the $\phi \to -\phi$ symmetry is satisfied at all orders in the EFT power counting, and n_d indicates the numbers of operators contributing to the EFT Lagrangian at mass-dimension d. Here for simplicity we omit the counterterms required to render the theory well-defined in the presence of loop corrections, it is always understood that these counterterms are present.

While the procedure leading to the bottom up construction of the EFT Lagrangian Eq. (3.1) appears to the straightforward, a common occurrence is that not all operators entering the expansion at any given order in δ are independent from each other. In addition, other operators do not contribute to physical observables. One therefore needs to ensure that, at any order in the power counting, the operator basis chosen is not redundant and does not contain spurious operators that do not affect physical observables. We describe next how to achieve this goal.

A non-redundant operator basis. As just mentioned, ideally at every mass dimension the set of operators $\{\mathcal{O}_i^{(d)}\}$ define an operator basis, that is, a set of operators which are

- *complete*: any other d-dimensional operator can be expressed by means of a combination of the given basis operators.
- *non-redundant*: all the elements of the operator basis are independent from each other and cannot be related by means of linear combinations or by usage of the equations of motion.

A redundant basis should be avoided since in such case some Wilson coefficients in the EFT will remain under-constrained and hence some physical observables will be ill-defined. Redundancies can arise do to relations induced by the EoM (equations of motion). A redundant basis should be avoided since in such case some Wilson coefficients in the EFT will remain under-constrained.

[1]Recall that in the UV theory the light field with mass m_ℓ is denoted by ϕ_ℓ. At tree level one has $m = m_\ell$ and the same for the field itself, but this is not true anymore once one includes higher-order corrections.

Another possible source of redundancies is the fact that operators which can be identified with total derivatives do not contribute to the action since they vanish in the integral

$$S_{\rm EFT} \propto \int d^4x \mathcal{L}_{\rm EFT} \,, \tag{3.2}$$

due to the fields vanishing at the integration boundaries $x^\mu \to \pm\infty$. These are operators that do not contribute to physical observables and that hence may be removed from the EFT Lagrangian.

Let us identify the types of operators that contribute to the scalar EFT Lagrangian defined by Eq. (3.1), focusing on the dimension-six operators, and ordering them by the number of derivatives that they contain. We will then determine which of those are really independent and which are redundant. These operators are the following:

- We can safely omit the $\Box\phi^4$ operator since it is a total derivative operator and hence would not contribute to the action (nor to physical observables). Recall that the d'Alembert operator (also known as "box" operator) is defined by

  $$\Box = \partial_\mu \partial^\mu = \eta^{\mu\nu}\partial_\mu\partial_\nu \tag{3.3}$$

  in terms of the Minkowski metric $\eta^{\mu\nu}$.

- There is a single operator without derivatives: ϕ^6, corresponding to a local six-point interaction.

- Two operators contain two derivatives, namely $\phi^3\Box\phi$ and $\phi^2\left(\partial_\mu\phi\right)\left(\partial^\mu\phi\right)$.

- Four operators contain four derivatives: $\phi\Box^2\phi$, $\left(\partial_\mu\phi\right)\left(\partial^\mu\Box\phi\right)$, $\left(\Box\phi\right)^2$, and $\left(\partial_\mu\partial_\nu\phi\right)\left(\partial^\mu\partial^\nu\phi\right)$.

so it would seem that we have seven independent operators that contribute to Eq. (3.1) at dimension-6 in the EFT expansion. You can convince yourself using dimensional analysis that we cannot add any other operator to the members of the $d = 6$ category. Applying the same reasoning to dimension-8 operators, it is clear that the number n_d of operators contributing to the EFT expansion grows very fast with the mass-dimension d.

However, it can be demonstrated that six of these operators are redundant and we thus have a single independent operator contributing to $\mathcal{L}_{\rm EFT}$ at dimension six (namely, the local six-point interaction). To see this, use the total derivative rule (usual integration by parts identity) to get

$$\partial_\mu\left(\phi^3\partial^\mu\phi\right) = 3\phi^2\left(\partial_\mu\phi\right)\left(\partial^\mu\phi\right) + \phi^3\Box\phi \,, \tag{3.4}$$

and hence we can drop $\phi^2\left(\partial_\mu\phi\right)\left(\partial^2\mu\phi\right)$ from our operator basis in the following. Indeed, the $\phi^3\Box\phi$ and $\phi^2\left(\partial_\mu\phi\right)\left(\partial^\mu\phi\right)$ operators are related by a total derivative and hence they are not independent from each other.

Similar considerations apply for other three operators, as one can see by noting that one can write

$$\begin{aligned}
\partial_\mu\left(\phi\partial^\mu\Box\phi\right) &= \left(\partial_\mu\phi\right)\left(\partial^\mu\Box\phi\right) \,, &(3.5)\\
\partial_\mu\left(\partial^\mu\phi\Box\phi\right) &= \left(\Box\phi\right)^2 + \left(\partial_\mu\phi\right)\left(\partial^\mu\Box\phi\right) \,, &(3.6)\\
\partial_\nu\left(\partial_\mu\phi\partial^\mu\partial^\nu\phi\right) &= \left(\partial_\mu\partial_\nu\phi\right)\left(\partial^\mu\partial^\nu\phi\right) + \phi^2\left(\partial_\mu\phi\right)\left(\partial^\mu\phi\right) \,. &(3.7)
\end{aligned}$$

In other words, if two operators can be related by a total derivative, they are not independent and hence once of them can be removed from the Lagrangian.[2]

Therefore, after we account for the fact that total derivatives do not contribute to the action, we are left with only 3 independent operators at dimension-six, which we take to be (this choice is of course not unique):

$$\phi^6, \phi^3\Box\phi, \phi\Box^2\phi\,, \tag{3.8}$$

which means that, up to dimension six, our EFT Lagrangian Eq. (3.1) is given by

$$\mathcal{L}_{\rm EFT} = \frac{1}{2}\left(\partial_\mu\phi\right)^2 - \frac{1}{2}m_\phi^2\phi^2 - \frac{\lambda}{4!}\phi^4 + \frac{c_{0,6}^{(6)}}{\Lambda^2}\frac{\phi^6}{6!} + \frac{c_{2,4}^{(6)}}{\Lambda^2}\frac{1}{3!}\phi^3\Box\phi + \frac{c_{4,2}^{(6)}}{\Lambda^2}\phi\Box^2\phi\,, \tag{3.9}$$

where the Wilson coefficients are labelled by the number of fields and derivatives of the corresponding operators. Hence our EFT has, up to $d = 6$, two contact interactions (4-point and 6-point) and two derivative interactions.

Field redefinitions. But actually, we are not done, since two of the dimension-six operators listed in the EFT Lagrangian of Eq. (3.9) can be shown to be redundant if we use the equations of motion of the theory. Using the EoMs is a bit more subtle than the previous discussion, so let's start with an example.

Consider a non-linear field redefinition for our light scalar field of the form

$$\phi(x) \to \phi(x) + \frac{a}{\Lambda^2}\phi^3(x)\,, \tag{3.10}$$

and now substitute it into the EFT Lagrangian that we have written down in Eq. (3.9). After some algebra we find the result after this non-linear field redefinition is

$$\mathcal{L}_{\rm EFT} \to \mathcal{L}_{\rm EFT} - \frac{a}{\Lambda^2}\phi^3\left(\Box + m_\phi^2 + \frac{\lambda}{3!}\phi^2\right)\phi + \mathcal{O}\left(\Lambda^{-4}\right)\,, \tag{3.11}$$

which means that if we choose $a = c_{2,4}^{(6)}/3!$, we can cancel out the operator $\phi^3\Box\phi$ from the EFT Lagrangian. This implies that if we redefine our scalar field according to Eq. (3.10) we can remove one of three dimension-six operators from our EFT Lagrangian.

To motivate why this kind of field redefinition is allowed, we can use a path-integral argument. This argument starts with the path integral associated to the Lagrangian under consideration

$$\mathcal{Z}\left[J\right] = \int \mathcal{D}\phi \exp\left[i\int d^4x\left(\mathcal{L}\left[\phi(x)\right] + J(x)\phi(x)\right)\right] \tag{3.12}$$

where $J(x)$ is known as the source field. Within the path-integral formalism, Green's function (aka field correlators) are evaluating by taking derivatives of the path integral with respect to the source field, that is

$$\frac{1}{\mathcal{Z}[J]}\frac{(-i)^n\delta^n\mathcal{Z}[J]}{\delta J(x_1)\cdots\delta J(x_n)}\bigg|_{J=0} = G(x_1,\ldots,x_n) = \langle\phi(x_1)\cdots\phi(x_n)\rangle\,. \tag{3.13}$$

In this formalism, in general one can redefine the field ϕ provided that this redefinition merely modifies the integration variable used in the functional integral, and does not actually modify the result of the integration. For a general field redefinition of the form

$$\phi(x) \to \phi'(x) = F[\phi(x)]\,, \tag{3.14}$$

[2]The choice of which one to keep is of course arbitrary.

the path integral changes as follows

$$\begin{aligned} \mathcal{Z}[J] &= \int \mathcal{D}\phi' \exp\left[i\int d^4x \left(\mathcal{L}[\phi'(x)] + J(x)\phi'(x)\right)\right] \\ &= \int \mathcal{D}\phi \left(\det\frac{\delta\phi'}{\delta\phi}\right) \exp\left[i\int d^4x \left(\mathcal{L}[F[\phi(x)]] + J(x)F[\phi(x)]\right)\right] \end{aligned} \tag{3.15}$$

where in the second line of the equation we account for the Jacobian associated to the field redefinition.

A field redefinition of the form Eq. (3.14) is allowed provided that the associated Jacobian satisfies:

$$\det\frac{\delta\phi'}{\delta\phi} = 1\,. \tag{3.16}$$

Let us show that this is the case for the field redefinition we are interested in, taking into account that we work in dimensional regularisation. To demonstrate this, let us assume that the general field redefinition in Eq. (3.14) takes the form

$$\phi'(x) = \phi(x) + \frac{1}{\Lambda^p} f[\phi(x)]\,, \tag{3.17}$$

and then the functional derivatives of the Jacobian lead to

$$\frac{\delta\phi'}{\delta\phi} = \delta^{(n)}(x-y) + \frac{1}{\Lambda^p} f'[\phi(x)]\delta^{(n)}(x-y)\,. \tag{3.18}$$

In order to evaluate the determinant, we can introduce Grassmann fields $c(x), \bar{c}(x)$ according to the nice formula

$$\det\frac{\delta\phi'}{\delta\phi} = \int\mathcal{D}\bar{c}\int\mathcal{D}c \exp\left[i\int d^n x\int d^n y\left(\bar{c}(x) i\frac{\delta\phi'}{\delta\phi}c(y)\right)\right] \tag{3.19}$$

where

$$i\int d^n x\int d^n y\left(\bar{c}(x) i\frac{\delta\phi'}{\delta\phi}c(y)\right) = -\int d^n x\left(\bar{c}(x)\left(1+\frac{1}{\Lambda^p}f'[\phi(x)]\right)c(x)\right)\,. \tag{3.20}$$

If now we treat the term suppressed by Λ^{-p} as a perturbation, we note that the $c-\bar{c}$ propagator is constant and hence all integrals are scaleless in dimensional regularization. Therefore $\exp[0]=1$ and as we wanted to demonstrate the Jacobian associated to the field redefinition is unity. This justifies our choice of field redefinition.

After this brief excursion into the path integral formalism, we can go back to our EFT discussion. Our argument indicates that field redefinitions of the form

$$\phi \to \phi + \frac{1}{\Lambda^p} f[\phi]\,, \quad p \geq 1\,, \tag{3.21}$$

are allowed, and this kind of redefinitions change the EFT Lagrangian into

$$\mathcal{L}_{\rm EFT} \to \mathcal{L}_{\rm EFT} - \frac{1}{\Lambda^p} f[\phi]\left(\Box + m_\phi^2 + \frac{\lambda}{3!}\phi^2\right)\phi \tag{3.22}$$

where the term in parenthesis can be identified with the Equations of Motion of the theory:

$$\text{EoM}: \left(\Box + m_\phi^2 + \frac{\lambda}{3!}\phi^2\right)\phi = 0\,. \tag{3.23}$$

Recall that the classical equations of motion of a relativistic field theory are given by the Euler-Lagrange equations derived from the least-action principle:

$$\frac{\delta S}{\delta \phi} = \frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \left(\frac{\delta \mathcal{L}}{\partial(\partial_\mu \phi)} \right) = 0 \,. \tag{3.24}$$

In the case in which we set to zero the four-point vertex, $\lambda = 0$, Eq. (3.23) is nothing but the Klein-Gordon equation for a free scalar field. Note also that since the field redefinition Eq. (3.21) is suppressed by powers of the high-scale Λ, it is formally higher order in the EFT expansion and does not modify the EoM of the low-energy theory (treating all higher-dimensional operators as corrections).

We can exploit the property that field redefinitions such as Eq. (3.21) are allowed and lead to a shift in the Lagrangian of the form of Eq. (3.22) and apply two of such field redefinitions such that we are only left with ϕ^6 as the only non-redundant dimension-six operator in our EFT Lagrangian. The starting point is the set of three dimension-six operators identified above, that survive after having eliminated all redundant terms which are related by total derivatives:

$$\frac{c^{(6)}_{0,6}}{\Lambda^2} \frac{\phi^6}{6!} + \frac{c^{(6)}_{2,4}}{\Lambda^2} \frac{1}{3!} \phi^3 \Box \phi + \frac{c^{(6)}_{4,2}}{\Lambda^2} \phi \Box^2 \phi \,. \tag{3.25}$$

First, we choose a field redefinition such that $f[\phi] = a\phi^3$, with $a = c^{(6)}_{2,4}/3!$. This transformation removes the second term in Eq. (3.25). This also implies that the four-point and six-point interactions change as:

$$\lambda \to \lambda - a \frac{m_\phi^2}{\Lambda^2} \,, \qquad c^{(6)}_{0,6} \to c^{(6)}_{0,6} - \frac{a\lambda}{\Lambda^2} \frac{6!}{3!} \,. \tag{3.26}$$

For the second field redefinition we use $f[\phi] = b\Box\phi$ with $b = c^{(6)}_{4,2}$, which then removes the third term in Eq. (3.25). The consequence of this field redefinition is a shift in the kinetic term

$$\frac{1}{2} \phi \Box \phi \to \left(1 - \frac{2b m_\phi^2}{\Lambda^2} \right) \frac{1}{2} \phi \Box \phi \tag{3.27}$$

and hence an additional field constant rescaling. Also the coefficient of the $\phi^3 \Box \phi$ operator gets modified, but this operator is anyway removed so we can ignore this effect. Crucially, the effects of these field redefinitions scale like Λ^{-p} and hence they vanish at leading order in the EFT expansion.

Therefore, we conclude that after removing operators that are made redundant by the EoMs and those that can be mapped to total derivatives, up to dimension-six the Lagrangian of our toy scalar EFT admits the following form

$$\mathcal{L}_{\rm EFT} = \frac{1}{2} \left(\partial_\mu \phi \right)^2 - \frac{1}{2} m_\phi^2 \phi^2 - \frac{\lambda}{4!} \phi^4 + \frac{c^{(6)}_{0,6}}{\Lambda^2} \frac{\phi^6}{6!} + \mathcal{O}\left(\Lambda^{-4} \right) \,, \tag{3.28}$$

with hence a single non-trivial dimension-six operator, namely the local six-point interaction.

This discussion highlights the fact that in general it is non-trivial to find complete, non-redundant basis for bottom-up constructions of EFTs: several effects need to be taken into account to ensure that the resulting EFT can deliver meaningful physical predictions for physical observables. Else, the EFT cannot be used to compare with the data, and similar problems would arise when matching to UV theories. Indeed, in the case in which $\mathcal{L}_{\rm UV}$ is the two-field model of Eq. (2.27), tree-level matching only generates the $\propto \phi^6$ interaction, hence this matching would also indicate that the other dimension-six operators entering the bottom up construction of the EFT Lagrangian Eq. (3.1) are physically irrelevant.

3.2 One-loop matching in scalar EFTs

The discussion that we carried out in the previous lecture concerning EFT matching was restricted to tree-level matching, which only involves Born amplitudes. Accounting for perturbative loop corrections to the matching relations is more subtle, as we discuss in this lecture. One-loop matching brings in qualitatively new effects in the relation between $\mathcal{L}_{\rm UV}$ and $\mathcal{L}_{\rm EFT}$ that are absent from the tree-level matching picture.

Once we consider loop integrals within a renormalized QFT, one has to take into account that:

- Loop integrals may contain UV divergences, which are cancelled by the counterterms. UV divergences will be different in the EFT and in the UV, since the two theories are different degrees of freedom at high energies.
- Loop integrals in the UV theory may have associated branch cuts involving light fields: these will be the same in the EFT and in the UV theory, since they correspond to common IR physics.
- Loop integrals in the UV theory have associated branch cuts involving heavy fields: these will be absent from the EFT, since such cuts are related to the UV scale and in the EFT there are no internal lines involving heavy particles.

A reminder that branch cuts arise in loop diagrams in QFTs whenever intermediate particles can go on shell.

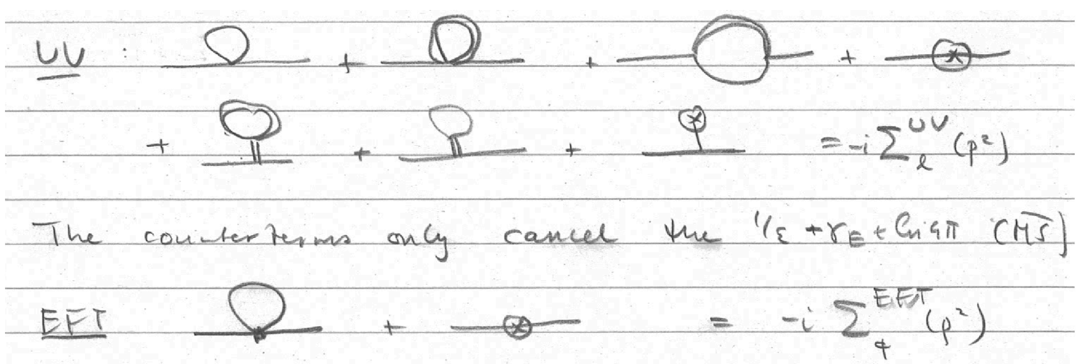

Figure 3.1: Feynman diagrams contributing to the one-loop calculation of the two-point function (self-energy) for the light scalar field in the UV theory (top panel) and in the EFT (bottom panel). The counterterms required to cancel the UV divergences in both theories are also indicated.

To illustrate how loop corrections affect matching in our two-particle scalar EFT, first of all let's consider the calculation of the self-energy (two-point function) of the light scalar. Again, we are working with the UV Lagrangian defined in Eq. (2.27) and composed by one light (ϕ_ℓ) and one heavy (Φ) scalar field. Fig. 3.1 shows the Feynman diagrams contributing to the one-loop calculation of the two-point function (self-energy) for the light scalar field in the UV theory (top panel) and in the EFT (bottom panel). The counterterms required to cancel the UV divergences in both theories are also indicated.

The calculation of these self-energy diagrams for the light scalar self-energy yields in the UV

$$\text{UV} : -i\Sigma_\ell^{\rm UV}(p^2)\,, \tag{3.29}$$

where the counterterms only cancel the $1/\epsilon + \gamma_E + \ln 4\pi$, and then in the EFT

$$\text{EFT} : -i\Sigma_\phi^{\rm EFT}(p^2)\,. \tag{3.30}$$

In order to determine the physical (pole) mass m_p after taking into account one-loop corrections, we take the inverse propagator defined in general as

$$p^2 - m^2 - \Sigma(p^2, m^2) \equiv C(p^2 - m_p^2)\,, \tag{3.31}$$

and then impose the on-shell condition

$$m_p^2 - m^2 - \Sigma(m_p^2, m^2) = 0\,, \tag{3.32}$$

$$m_p^2 = m^2 + \Sigma(m_p^2, m^2)\,. \tag{3.33}$$

Imposing that the pole mass is the same in the EFT and in the UV theory (since it is an observable present at low energies and hence must coincide in the EFT and in the UV) yields

$$m_p^2 = m_\ell^2(\mu) + \Sigma_\ell^{\rm UV}(m^2) = m_\phi^2(\mu) + \Sigma_\phi^{\rm EFT}(m_\phi^2)\,, \tag{3.34}$$

where note that the renormalized EFT and UV masses depend on the renormalization scale μ which enters via the one-loop corrections, and therefore we obtain the following relation

$$m_\phi^2(\mu^2) = m_\ell^2(\mu^2) + \Sigma_\ell^{\rm UV}(m_\ell^2, \mu^2) - \Sigma_\phi^{\rm EFT}(m_\phi^2, \mu^2)\,, \tag{3.35}$$

and as we said all masses are renormalized, since the counterterms are already included.

By evaluating the associated Feynman diagrams in the UV theory and in the EFT, one gets for the latter the following result for the self-energy

$$\Sigma_\phi^{\rm EFT}(m_\phi^2, \mu^2) = -\frac{\lambda m_\phi^2}{32\pi^2}\left(1 - \ln\frac{m_\phi^2}{\mu^2}\right)\,, \tag{3.36}$$

while for the UV theory, already expanded in the small parameter m_ℓ^2/M^2, one has

$$\begin{aligned}\Sigma_\ell^{\rm UV}(m_\ell^2, \mu^2) \;&=\; -\frac{1}{32\pi^2}\left[g_\phi m_\ell^2 + (2\tilde{g}_\phi^2 - \tilde{g}_\Phi\tilde{g}_\phi + g_{\phi\Phi})M^2 + \frac{10}{3}\tilde{g}_\phi^2\frac{m_\ell^4}{M^2}\right]\\ &+\; \frac{1}{32\pi^2}\left[\cdots\right] m_\ell^2 \ln\frac{m_\ell^2}{\mu^2} + \frac{1}{32\pi^2}\left[\cdots\right]\ln\frac{M^2}{\mu^2}\,,\end{aligned} \tag{3.37}$$

which depends on various UV parameters, in particular the heavy field couplings and mass M. Recall that we know from the tree level matching of this theory, Eq. (2.39), that $\lambda = g_\phi +$ higher orders and hence for the accuracy of this calculation we can take $\lambda = g_\phi$. Likewise, we can use that $m_\ell = m_\phi$ up to higher orders in the EFT expansion. Therefore, matching the two-point functions at the heavy mass scale $\mu^2 = M^2$ (such that the logarithms $\ln(M^2/\mu^2) \to 0$ vanish) results into the following relation

$$m_\phi^2(M) = m_\ell^2(M) - \frac{1}{32\pi^2}\left[3\tilde{g}_\phi^2 m_\ell^2 + (2\tilde{g}_\phi^2 - \tilde{g}_\Phi\tilde{g}_\phi + g_{\phi\Phi})M^2 + \frac{22}{3}\tilde{g}_\phi^2\frac{m_\ell^4}{M^2}\right]\,, \tag{3.38}$$

where as we know if we set the UV couplings to zero (except g_ϕ) then the UV and EFT light mass is the same and also does not depend on the matching scale. This makes sense: g_ϕ is present both in the UV and in the EFT, and hence its effects in the IR will be shared in the two calculations.

The hierarchy problem in scalar QFT. The key feature of this result is that the relation between the UV mass $m_\ell^2(M)$ and the IR mass $m_\phi^2(M)$ depends strongly on the UV physics scale M:

$$m_\phi^2(M) \simeq m_\ell^2(M) - \frac{1}{32\pi^2}\left[M^2(\cdots)\right] , \tag{3.39}$$

which implies that to keep the renormalised mass of the light scalar field in the IR, $m_\phi^2(M)$, sufficiently small, an in particular to ensure that $m_\phi^2(M) \ll M^2$, in the presence of loop corrections in the limit $M \gg m$ (that is, a large hierarchy of scales) one needs to introduce *fine-tuning* between the renormalized masses m_ℓ, M of the UV theory.

Indeed, a fine-tuned choice of $m_\ell(M)$ and M in the UV theory would cancel the quadratic dependence in Eq. (3.38) and reduce it to a much weaker logarithmic dependence. This fine tuning is technically possible, but theoretically not very elegant. This is nothing but the extremely famous fine-tuning problem of QFTs involving scalars, which drives the so-called "Higgs hierarchy problem" in the Standard Model of particle physics. This is a generic feature of QFTs with fundamental scalars: unless ome symmetry protects their mass in the IR, loop corrections tend to make the mass of any light scalar fields unstable and become very large, towards the UV limit.

3.3 Decoupling in EFTs

Using similar methods, one could carry out the one-loop matching between our scalar EFT and UV theories to determine the values of λ and $c_{0,6}^{(6)}$ which account for one-loop corrections in the UV coupling constants, extending the tree level matching results that we derived in the previous lecture. For this, one needs to repeat the calculation we presented at the tree level (namely the calculation of the four-point and six point amplitudes) but now considering also one loop corrections both from the EFT and the UV sides.

Instead, we will proceed here to highlight a somewhat different approach. The idea is to change the renormalization scheme such that all diagrams with internal heavy quark lines can be omitted at leading power in the EFT expansion. This choice results in the method known as *decoupling*, in that heavy physics is decoupled from calculations in the IR also in the present of one-loop corrections. Let's show how this works with an example. Consider the one-loop correction to the four-point function in the UV theory shown in Fig. 3.2 (together with the corresponding counterterm). This loop diagram yields

$$= -\frac{1}{32\pi^2}\int_0^1 dx \left(\frac{1}{\epsilon} - \gamma_E + \ln 4\pi - \ln \frac{\left(M^2 - x(1-x)(p_1 + p_2^2)^2\right)}{\mu^2}\right) , \tag{3.40}$$

with dependence on M^2 entering via the heavy scalar lines, with the corresponding counterterms giving

$$\frac{1}{32\pi^2}\frac{1}{\epsilon} \qquad (\mathrm{MS-scheme}) \tag{3.41}$$

$$\frac{1}{32\pi^2}\left(\frac{1}{\epsilon} - \gamma_E + \ln 4\pi\right) \qquad (\overline{\mathrm{MS}}-\mathrm{scheme}) \tag{3.42}$$

The CWZ approach to manifest decoupling is based on subtracting the UV divergence at zero momentum ($p = p_1 + p_2 = 0$), and hence the counterterm that defines the decoupling scheme is given by

$$\frac{1}{32\pi^2}\left(\frac{1}{\epsilon} - \gamma_E + \ln 4\pi - \int_0^1 dx \ln \frac{M^2}{\mu^2}\right) \qquad (\mathrm{decoupling-scheme}) . \tag{3.43}$$

The main difference is that the MS and $\overline{\text{MS}}$ renormalization schemes leave the dependence on the heavy field mass $\ln M^2/\mu^2$ in the renormalized amplitude, where instead the CWZ (decoupling) scheme leads to the following result for the subtracted one-loop diagram

$$-\frac{1}{32\pi^2}\int_0^1 dx \left(\ln\frac{\left(M^2 - x(1-x)(p_1+p_2^2)^2\right)}{\mu^2} - \ln\frac{M^2}{\mu^2}\right) = -\frac{1}{32\pi^2}\int_0^1 dx \left(\ln\frac{\left(1 - x(1-x)s/M^2\right)}{\mu^2}\right) .$$

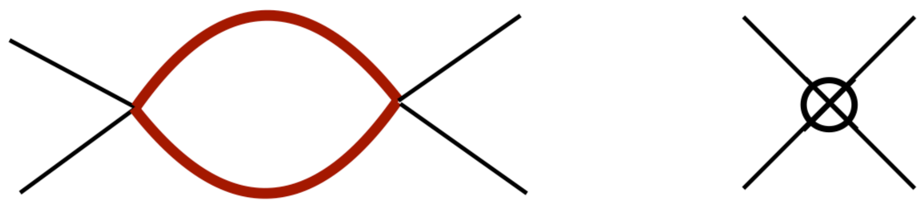

Figure 3.2: One-loop correction to the four-point amplitude $\mathcal{A}_{\text{UV}}\left(\phi_\ell\phi_\ell \to \phi_\ell\phi_\ell\right)$ and its corresponding counterterm. The thick red lines indicates heavy field propagators.

One may think that the situation in Eq. (3.44) is not much better, since there is still dependence on the heavy scalar M. However, the dependence on M is now suppressed in the IR limit $s \ll M^2$. Indeed upon expanding in the EFT power counting parameter s/M^2 we obtain for this one-loop correction to the four-point amplitude

$$\mathcal{A}_{\text{UV}}\left(\phi_\ell\phi_\ell \to \phi_\ell\phi_\ell\right) \supset -\frac{1}{32\pi^2}\int_0^1 dx \left(-\frac{s}{M^2}x(1-x) + \mathcal{O}\left(\frac{s}{M^2}\right)^2\right) , \tag{3.44}$$

which exhibits manifest decoupling, in the sense of loop corrections involving heavy lines lead to a vanishing contribution in the limit $s \ll M^2$ without the need to introduce fine tuning.

This decoupling scheme for renormalization sizable facilitates some matching calculations in EFTs (as well as in regular QFTs, such as for example in the case of heavy quark mass effects in QCD) since it manifestly conserves the natural hierarchy of mass scales also in the presence of higher order corrections.

3.4 The method of regions

Another strategy to facilitate matching calculations in EFTs is the following. Let's consider the matching in $\mathcal{A}_{\text{UV}} - \mathcal{A}_{\text{EFT}}$. The amplitude $\mathcal{A}_{\text{UV}}$ will exhibit in general both poles and branch cuts for both heavy and light fields. The amplitude $\mathcal{A}_{\text{EFT}}$ will exhibit only poles and branch cuts associated to the light fields, and actually these must the same as in its UV counterpart. Let's consider the following one loop correction to the propagator in the UV theory. Using standard matching techniques one obtains

$$I^{\text{UV}} + I^{\text{UV}}_{\text{ct}} = \frac{-i}{16\pi^2}\left(1 - \frac{1}{M^2 - m^2}\left(M^2\ln\frac{M^2}{\mu^2} - m^2\ln\frac{m^2}{\mu^2}\right)\right) . \tag{3.45}$$

The same calculation in the EFT, shrinking the heavy scalar line to a point and adding the corresponding counterterms, yields the expansion

$$\frac{i}{q^2 - M^2} = -i\left(\frac{1}{M^2} - \frac{q^2}{M^4} + \dots\right) \tag{3.46}$$

the following result

$$\begin{aligned} I^{\rm EFT} &= \mu^{2\epsilon}\int \frac{d^n q}{(2\pi)^n}\frac{1}{M^2}\left(\frac{1}{M^2} - \frac{q^2}{M^4} + \dots\right)\frac{1}{q^2 - m^2} \\ &= \frac{i}{16\pi^2}\frac{m^2}{M^2}\left(1 + \frac{m^2}{M^2} + \dots\right)\left[\frac{1}{\epsilon} + 1 - \gamma_E + \ln 4\pi - \ln\frac{m^2}{\mu^2}\right] \end{aligned} \tag{3.47}$$

where as usual the $(1/\epsilon - \gamma_E + \ln 4\pi)$ terms cancel out with the corresponding counterterm.

We can now evaluate the following "difference" between the UV and EFT integrals to obtain

$$I^{\rm EFT}_{\rm match} \equiv \left(I^{\rm UV} + I^{\rm UV}_{\rm ct}\right) - \left(I^{\rm EFT} + I^{\rm EFT}_{\rm ct}\right) = \frac{-i}{16\pi^2}\left(1 + \frac{m^2}{M^2} + \dots\right)\left(1 - \ln\frac{M^2}{\mu^2}\right). \tag{3.48}$$

The trick now is the following. Since the analytic behavior (poles and branch cuts) for the light fields is the same in the EFT and in the UV theory, we can expand in the integrand

$$\frac{1}{q^2 - m^2} = \frac{1}{q^2}\left(1 + \frac{m^2}{q^2} + \cdots\right) \tag{3.49}$$

so that the loop integral becomes

$$I^{\rm EFT}_{\rm exp} = \mu^{2\epsilon}\int \frac{d^n q}{(2\pi)^n}\frac{1}{M^2}\left(1 + \frac{q^2}{M^2} + \frac{q^4}{M^4}\right)\frac{1}{q^2}\left(1 + \frac{m^2}{q^2} + \cdots\right) \tag{3.50}$$

but now we notice that each of the individual terms of this integral is scaleless and hence they all vanish. This means that we can write

$$I^{\rm UV}_{\rm match} = \left(I^{\rm UV} + I^{\rm UV}_{\rm ct}\right) - \left(I^{\rm EFT} + I^{\rm EFT}_{\rm ct}\right) = \left(I^{\rm UV}_{\rm exp} + I^{\rm UV}_{\rm ct}\right) - I^{\rm EFT}_{\rm ct}, \tag{3.51}$$

where

$$I^{\rm UV}_{\rm exp} = \frac{-i}{16\pi^2}\frac{m^2}{M^2}\left(1 + \frac{m^2}{M^2} + \cdots\right)\left[\frac{1}{\epsilon} - \gamma_E + \ln 4\pi + 1 - \ln\frac{M^2}{\mu^2}\right], \tag{3.52}$$

where now the $1/\epsilon$ pole is associated to IR physics. Then we find that the full UV integral can be expressed in the rather neat way

$$I^{\rm UV} = I^{\rm UV}_{\rm exp} + I^{\rm EFT}, \tag{3.53}$$

where:

- $I^{\rm UV}_{\rm exp}$ contains the contribution from the *hard region* sensitive to UV physics, namely $q^2 \sim M^2$ with $q^2 \gg m^2$.
- $I^{\rm EFT}$ contains the contribution from the *soft region* sensitive to IR / low energy physics, namely $q^2 \sim m^2$, $q^2 \ll M^2$.

This result is at the core of what is called the *Method of Regions*: being able to decompose a given (tree or loop-level) amplitude into separate contributions, each of them associated to a different region of energy and momenta. This analysis greatly facilitates the calculation of some loop integrals, since the method of region allows to cleanly separate regions sensitive to UV and to IR physics.

3.5 Summary and outlook

To summarize the main points considered in Lectures 2 and 3:

- EFTs can be either constructed from the top-down (matching to UV Lagrangians) or from the bottom up (starting from specific field content and symmetries).
- EFTs can be matched to UV physics either at tree level or including loop corrections. For scalar fields, loop corrections introduce fine tuning problems between IR and UV physics.
- In the bottom up construction of EFTs, it is essential to discard operators which are redundant or which do not have implications for physical observables.

In the rest of the course, we discuss how these ideas are applied to more realistic QFTs, such as the low-energy weak effective theory, the SM Effective Field Theory, and chiral perturbation theory.

Effective Field Theory

Current version: **June 13, 2024**

4 Lecture 4: Weak EFT part I

Learning Goals

- Derive the Fermi theory of weak interactions and connect it with modern EFT viewpoint.
- Construct the low-energy EFT of the SM well below the electroweak scale, both bottom up and top down.
- Determine how the effects of operator running and mixing modify the predictions of an EFT.

In the previous two lectures we have demonstrated the main principles of the Effective Field Theory paradigm in the case of a toy model composed by a light and a heavy scalar fields. We have learned how to build the corresponding EFT both using bottom-up and top-down methods, and how to carry out the matching to the UV physics both at tree-level and at the one-loop level.

We now apply these ideas to (mostly) the electroweak sector of the SM. First of all we consider Fermi theory of beta decay, which is perhaps the first historical example of an EFT in particle physics. Then we tackle the whole SM, and determine what is the low energy limit (where low energy means energies well below the m_W mass), discuss its phenomenological implications, and describe what happens when the EFT operators mix among them in the presence of loop corrections.

4.1 The role of the electroweak scale and Fermi Theory

The Standard Model of particle physics is composed of particles spanning a very broad mass range, from the electron ($m_e \sim 0.5$ MeV) all the way up to the top quark ($m_t \sim 175$ GeV). In particular, the electroweak gauge bosons ($W^{\pm}, Z$), which acquire they masses via the electroweak symmetry breaking process, have masses of the order $m_V = \mathcal{O}(v)$, with $v = 246$ GeV being the vacuum expectation value (vev) of the Higgs boson. Specifically, the W boson mass is given by

$$v = \frac{2m_W}{g_{\rm EW}}\,, \tag{4.1}$$

with $g_{\rm EW}$ being the weak isospin coupling. Once one includes numerical prefactors, we find that theory predicts $m_W \sim 80$ GeV and $m_Z \sim 91$ GeV in agreement with the experiment.

From the same considerations we used in the case of the two-scalar toy model of the previous lectures, it should be clear that for processes involving the weak interactions, but where the typical momentum transfers satisfy $p \ll m_W$, the weak gauge bosons will never appear as explicit degrees of freedom in the calculation of scattering amplitudes (neither as internal nor as external lines) and hence can be "integrated out" from the low energy theory. It then makes sense to consider an EFT of the SM where the electroweak gauge bosons as well as other heavy fields (for consistency) are integrated out.

In this lecture we will study the so-called **Weak EFT** (WET), which is the effective field theory resulting from starting with the SM and then integrating out the "heavy" fields at the electroweak scale and above, that is, degrees of freedom which decouple in processs incolving momentum transfers $p \ll m_W$. This EFT is also known as the Low-energy EFT (LEFT), depending on the convention. This means that the particle content in the "UV theory" (the SM) and in the Weak EFT (WET) will be

$$\text{SM}: \quad u, d, s, c, b, t, e, \mu, \tau, \nu_e, \nu_\mu, \nu_\tau, W, Z, h, g, \gamma \tag{4.2}$$

$$\text{Weak EFT}: \quad u, d, s, c, b, , e, \mu, \tau, \nu_e, \nu_\mu, \nu_\tau, g, \gamma \tag{4.3}$$

As in the case of the scalar EFT discussed in the previous lectures, integrating out the heavy fields does not mean that the effects of the heavy particles disappear from the EFT: they are present in the form of modified interactions and effective couplings between the low-energy degrees of freedom.

The paradigmatic application of the WET is the Fermi theory of the weak interactions. Using the same original application that motivated Fermi theory, let us consider the decay of the muon in the SM (Fig. 4.1), which is described by the Feynman diagram shown in Fig. 4.1. In the rest frame of the muon, the typical momentum transfers involved in this diagram will be

$$p_W^\mu \sim \mathcal{O}(m_\mu) \ll m_W \,, \tag{4.4}$$

which implies that we can describe this process by means of an EFT where the intermediate W boson has been integrated out. Let us see what are the results of integrating out this "heavy" intermediate particle.

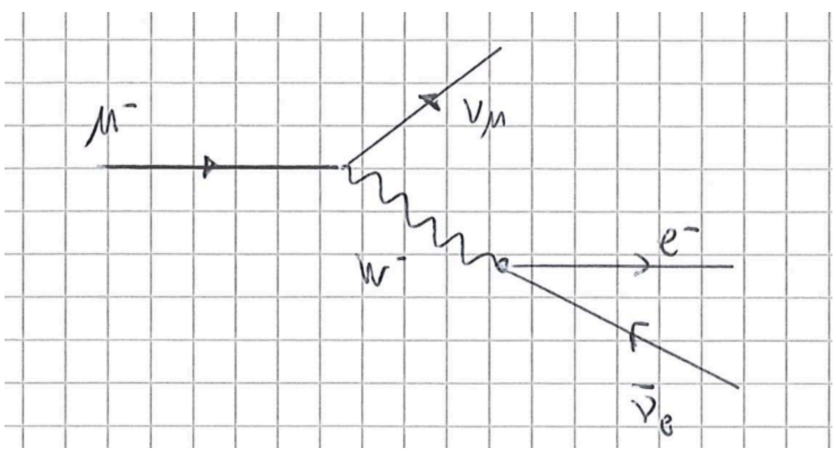

Figure 4.1: Muon decay in the SM at tree level.

In the SM, the W-boson is characterized by *chiral interactions*: this means that it couples differently to left-handed and right-handed fermions. Specifically, the SM W-boson has vanishing interaction with right-handed fermions, so its interactions are maximally chiral. The relevant part of the SM Lagrangian that describes these interactions is

$$\mathcal{L}_{\text{SM}} \supset -\frac{g_2}{\sqrt{2}} W_\mu^+ j_W^\mu + \text{h.c.} \,, \tag{4.5}$$

where the source current to which the $W^{\pm}$ boson couples is

$$j_W^{\mu} = \sum_{\ell=e,\mu,\tau} \bar{\nu}_\ell \gamma^\mu P_L \ell + \text{quark current}\,, \tag{4.6}$$

where:

- g_2 is the $\mathrm{SU}(2)_L$ coupling.
- The charged-fermion field appears together with a left-handed projector

$$P_L \equiv \frac{1}{2}\left(1-\gamma_5\right) P_T \equiv \frac{1}{2}\left(1+\gamma_5\right)\,, \tag{4.7}$$

  which removes the right-handed component.
- It is not necessary to apply a P_L projector to the neutrino field since neutrinos (in the SM) only have left-handed spinor components.
- We ignore here the quark current, which have associated some subtleties associated to CKM mixing (which furthermore are not relevant for the case at hand).

Using the interaction vertex described by Eq. (4.5), we can compute the tree-level amplitude for the muon decay in the SM:

$$\begin{aligned} i\mathcal{M}(\mu^-(p_1) \to \nu_\mu(p_2) + e^-(p_3) + \bar{\nu}_e(p_4)) &= \left(-i\frac{g_2}{\sqrt{2}}\right)^2 \left[\bar{u}_{\nu_\mu}(p_2)\gamma^\mu P_L u_\mu(p_1)\right] \\ &\times \left(\frac{-i}{p^2-m_W^2}\left(g_{\mu\nu} - \frac{p_\mu p_\nu}{m_W^2}\right)\right) \times \left[\bar{u}_\ell(p_3)\gamma^\nu P_L u_{\nu_e}(p_4)\right]\,, \end{aligned} \tag{4.8}$$

where we work in the unitary gauge, and we have defined $p^\mu \equiv p_1^\mu - p_2^\mu$ as the momentum flowing across the W^- line (see also Fig. 4.1).

If we were carrying out the calculation in the SM, we would now proceed with the usual Dirac algebra manipulations to simplify this expression. But since our goal is to work in the WET limit where the W boson can be integrated out, we can already now simplify this amplitude and reduce our calculational burden. For this, we recall that we work in the limit defined by Eq. (4.4) and hence the small parameter that defines this EFT expansion will be

$$\delta \equiv \frac{p^2}{m_W^2} \ll 1 \qquad (p \sim \mathcal{O}(m_\mu))\,, \tag{4.9}$$

and expanding in a power series in δ the W boson propagator one gets

$$\frac{1}{p^2-m_W^2} = -\frac{1}{m_W^2}\left(1 - \frac{p^2}{m_W^2} + \left(\frac{p^2}{m_W^2}\right)^2 + \cdots\right)\,. \tag{4.10}$$

In particular, one notes that at leading order in the EFT expansion in δ there is no dependence at all on the W-boson momentum p: this means that muon decay in the WET must be described by a *contact interaction* (hence without involving derivatives).

Keeping only the first non-trivial term in the WET expansion in the calculation of the matrix element Eq. (4.8), the muon decay amplitude simplifies to

$$i\mathcal{M}(\mu^- \to \nu_\mu + e^- + \bar{\nu}_e) = -i\frac{g_2^2}{2m_W^2}\left[\bar{u}_{\nu_\mu}(p_2)\gamma^\mu P_L u_\mu(p_1)\right]\left[\bar{u}_\ell(p_3)\gamma^\nu P_L u_{\nu_e}(p_4)\right] + \mathcal{O}\left(p^2/m_W^2\right)\,, \tag{4.11}$$

which as mentioned above does not contain (at this order) any dependence with the W-boson momentum p. Note also that in the limit $m_W \to \infty$ then the matrix element vanishes $\mathcal{M} \to 0$: hence existence of muon decay implies the existence of a mediator particle, with details about muon decay kinematics providing information on the properties of such heavy mediator.

The muon decay amplitude in the SM with the simplification implemented in Eq. (4.11) can be reproduced if we consider a WET with the active degrees of freedom listed in Eq. (4.3) which includes a contact interaction of the form

$$\mathcal{L}_{\mathrm{WET}} \supset \frac{c}{\Lambda^2} \left(\bar{\nu}_\mu \gamma^\mu P_L \mu\right) \left(\bar{e} \gamma_\mu P_L \nu_e\right) + \text{h.c.}\,, \tag{4.12}$$

where c is a Wilson coefficient and Λ is a large energy scale, related to the limit of validity of the EFT. As we have seen in the case of the toy scalar theory, the combination c/Λ^2 can be determined by matching to the corresponding UV theory, in this case the SM. For this specific application, tree-level matching is trivial, since the Lagrangian Eq. (4.12) directly leads to an amplitude of the form Eq. (4.11), and hence we determine the Wilson coefficient to be

$$\frac{c}{\Lambda^2} = -\frac{g_2^2}{2m_W^2}\,, \tag{4.13}$$

reflecting that the strength of the effective four-fermion interaction arising in the WET after we integrate out the W boson remain dictated by the couplings and masses of the integrated-out fields: a measurement of the muon decay rate tells us about the mass scale of a much heavier particle, the W boson, even from low-energy measurements.

Predicting the weak scale. While the outcome of matching the SM to the WET is not particularly surprising, it is worth recalling that historically the four-fermion interaction Eq. (4.12) describing muon decay and known as *Fermi Theory* preceded the formulation of the SM. It is thus worth reflecting how the experimental evidence for Fermi theory as a valid description of muon decay provided valuable information on the value of the weak scale and hence on the masses of the (by then still unobserved) W, Z bosons. From the point of view of a bottom-up construction of the Fermi theory as an EFT, the WET Lagragian is expressed in terms of the Fermi constant G_F,

$$\mathcal{L}_{\mathrm{WET}} \supset -\frac{4G_F}{\sqrt{2}} \left(\bar{\nu}_\mu \gamma^\mu P_L \mu\right) \left(\bar{e} \gamma_\mu P_L \nu_e\right)\,, \tag{4.14}$$

where the dimensionful Fermi constant has a numerical value of

$$G_F = 1.16637 \times 10^{-5}\,\mathrm{GeV}^{-2} \tag{4.15}$$

and can be extracted by comparing the prediction of the muon decay width computed using the Lagrangian in Eq. (4.14),

$$\Gamma = \frac{G_F^2}{192\pi^3} m_\mu^5\,, \tag{4.16}$$

with the corresponding experimental value (since the muon mass can be determined by other means).

From dimensional analysis, one notes that indeed the existence of the four-fermion interaction Eq. (4.12) driving muon decay necessitates the existence of a high scale Λ where "new physics" become active degrees of freedom: the reason is that scattering amplitudes involving four-fermion interactions receive large corrections as $p \to \Lambda$ and eventually violate unitarity. Given the mass dimension of G_F, one expects that its relation with the "new physics" scale Λ is

$$G_F \propto \Lambda^{-2}\,, \tag{4.17}$$

assuming $\mathcal{O}(1)$ numerical coefficients. Assuming that interactions at the UV scale are perturbative, we can set an upper bound on the scale Λ, namely

$$\Lambda \leq G_F^{-1/2} \simeq 300 \text{ GeV}\,, \tag{4.18}$$

and therefore, the existence of muon decay mediated by a four-fermion interaction points out to the scale of new physics (in this case, the electroweak scale) being a few hundreds of GeV. We can compare this estimate with the exact result obtained from tree-level matching Eq. (4.13),

$$G_F = \frac{\sqrt{2}g_2^2}{8m_W^2} = \frac{1}{\sqrt{2}v^2} \qquad \Lambda^2 \leq v^2\sqrt{2}\,. \tag{4.19}$$

Historically, this prediction has been one of the main achievements of the EFT formalism, highlighting how quantum corrections provide information on short-distance dynamics also from measurements directly sensitive only to long-distance

4.2 Assembling the Weak Effective Theory

After this prelude with the Fermi theory, we are ready to assemble the WET using the same philosophy as in the case of the toy scalar model. A reminder that our goal is to formulate an EFT which only contains the degrees of freedom and symmetries of the SM relevant to describe the physics at $E \ll m_W$.

For such construction, the expansion parameter is $\delta \simeq p^2/m_W^2$, such that at $\mathcal{O}(\delta^0)$ (dimension-6) we have four-fermion operators, at $\mathcal{O}(\delta^1)$ (dimension-8) we have four-fermion operators with derivative couplings as well as six-fermion operators, and so on. When constructing the WET Lagrangian, we need to consider all possible operators that can be built using the light fields listed in Eq. (4.3) as degrees of freedom. In addition, the WET operators need to satisfy a number of symmetries, both local and global, and also both exact and accidental. In particular, we will impose that WET operators satisfy the following requirements:

- Lorentz invariance.
- The same gauge symmetries as the SM, but after electroweak symmetry breaking, which are those relevant at energy scales below the weak scale. That is, while the SM Lagrangian is invariant under the gauge group
$$\text{SU}(3)_c \otimes \text{SU}(2)_L \otimes \text{U}(1)_Y\,, \tag{4.20}$$
the WET Lagrangian is invariant under the subset that remains after electroweak symmetry breaking, namely
$$\text{SU}(3)_c \otimes \text{U}(1)_{\text{EM}}\,. \tag{4.21}$$
In particular, we note that the gauge symmetries required in the WET lead to vector-like interactions, unlike the chiral interactions of the weak sector of the SM. This does not mean that WET interactions will be always vector-like, but rather than chiral interactions are not demanded by any symmetry in the theory.
- We also impose the *accidental symmetries* of the SM, such as baryon number conservation and lepton number conservation. These are per se not needed for the WET (since they are not fundamental symmetries) but since operators that violate them are well constrained, for simplicity we also assume them in this case.[3]

[3]Operators that violate symmetries like LFN and BN are tightly constrained by many experiments and point to a very high scale, well above those probed at any present or future collider.

To summarize, the bottom-up construction of the WET is based on the following ingredients:

- The active degrees of freedom (particle content): $u, d, s, c, b, , e, \mu, \tau, \nu_e, \nu_\mu, \nu_\tau, g, \gamma$.
- The symmetries listed above, and in particular gauge symmetry $\mathrm{SU}(3)_c \otimes \mathrm{U}(1)_{\mathrm{EM}}$.
- The power counting:

$$\delta \simeq \frac{m^2}{v^2}, \frac{p^2}{v^2} \ll 1 \,, \tag{4.22}$$

  with m being one of the masses of the active degrees of freedom. One can also replace v by m_W in the power expansion, since the two quantities are parametrically the same.

At leading order in the EFT expansion, that is, in the limit $\delta \to 0$, the WET Lagrangian admits the particularly simple form

$$\mathcal{L}_{\mathrm{WET}} = \mathcal{L}_{\mathrm{QCD}} + \mathcal{L}_{\mathrm{EM}} + \mathcal{O}\left(\frac{1}{v^2}\right) \,, \tag{4.23}$$

with only the light fields participating. That is, all weakly interacting particles are fully decoupled (and in particular neutrinos are absent at this order). The fact that neutrinos are absent in the limit $\delta \to 0$ is expected since they are neutral under the gauge symmetries of the EFT, and hence can only arise for effective interactions mediated by the high-energy degrees of freedom that have been integrated out. For the same reason, at $\mathcal{O}\left(\delta^0\right)$ in the WET, muons are stable and do not decay (there is no interaction than can mediate such decay). Recall that in the Fermi theory, four-fermion interactions that mediate muon decay start at $\mathcal{O}(\delta)$.

To make this limit ($\delta \to 0$) more explicit we write the QCD and QED Lagrangians involving only the light degrees of freedom,

$$\mathcal{L}_{\mathrm{WET}} = \mathcal{L}_{\mathrm{QCD}} + \mathcal{L}_{\mathrm{EM}} + \mathcal{O}\left(\frac{1}{v^2}\right) = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{4}G^A_{\mu\mu}G^{A\mu\nu} + \theta_{\mathrm{QCD}}\frac{g^2}{32\pi^2}G^A_{\mu\mu}\widetilde{G}^{A\mu\nu} \tag{4.24}$$

$$+ \sum_{u,d,s,c,b,e,\mu,\tau} \bar{\psi} i \not{D} \psi - \left[\sum_{u,d,s,c,b,e,\mu,\tau} \bar{\psi}_R m_\psi \psi_L + \mathrm{h.c.}\right] \tag{4.25}$$

with the covariant derivative given by

$$D_\mu = \partial_\mu + igT^A G^A_\mu + ieQA_\mu \,, \tag{4.26}$$

including only the coupling to gluons (QCD) and to photons (QED). In the case that the accidential symmetries of the SM are disregarded, there would be additional terms at this order, in particular a Majorana mass term for the neutrinos violating lepton number conservation by $\Delta L = 2$.

This Lagrangian corresponds to the zero-th order contribution to the WET. The subsequent orders can be computed by adding all higher-order operators built upon the WET degrees of freedom and consistent with the assumed symmetries. That is, taking into account the power-counting expansion, we will have that the WET Lagrangian reads

$$\mathcal{L}_{\mathrm{WET}} = \mathcal{L}_{\mathrm{QCD}} + \mathcal{L}_{\mathrm{EM}} + \sum_{d\geq 5}\sum_{i=1}^{n_d} L_i^{(d)}\mathcal{O}_i^{(d)} \,, \tag{4.27}$$

where in this notation the Wilson coefficients are dimensionful and scale like

$$L_i^{(d)} \propto v^{4-d} \,, \qquad d \geq 5 \,. \tag{4.28}$$

It is interesting to note that at $\mathcal{O}\left(v^0\right)$, the WET is not a chiral theory but rather a vector theory, in that all interactions treat in the same way left-handed and right-handed fermions. Chiral interactions arise only at $\mathcal{O}\left(v^{-2}\right)$, with for example the four-fermion interactions that we saw in Fermi Theory treat differently left-handed from right-handed fermions. As mentioned above, the reason is that chiral interactions can be constructed even with purely vector gauge interactions once higher-dimensional operators are allowed.

4.3 The WET operator basis

Let us now discuss which kind of operators may be present in the WET Lagrangian given by Eq. (4.27). In the following, we denote field-strength tensors as $X = F_{\mu\nu}, G_{\mu\nu}$, and ψ indicates fermion fields.

The lowest non-trivial order of the WET is $d = 5$. At $d = 5$, we will have operators of the form $\mathcal{O} = \psi^2 X$, while at $d = 6$ power counting tells us that operators will be either for the form $\mathcal{O} = X^3$ (purely gauge fields) or $\mathcal{O} = \psi^4$ (the four-fermion operators present in Fermi Theory). Hence we see that the general WET contains several other operators beyond those that were considered by the original Fermi Theory (as expected, since this was limited to four-fermion interactions).

At dimension-6, In addition to operators of the form $\mathcal{O} = X^3$ and $\mathcal{O} = \psi^4$, one can consider other possible types of WET operators, for example, those containing derivatives. One could think for example of operators of the form $\mathcal{O} \sim (\partial X)^2$, which also have the right dimensions. However, you can convince yourself that this operator is *redundant* because it can be related to other operators that we are already considering by using the Equations of Motion of the lowest-order Lagrangian. Indeed, applying the equations of motion to the QCD and QED Lagrangians yields

$$\partial_\mu F^{\mu\nu} = e \sum_{\psi=u,d,\ell} \bar{\psi}\gamma^\nu Q \psi \,, \tag{4.29}$$

$$(D_\mu G^{\mu\nu})^A = g_s \sum_{\psi=u,d} \bar{\psi}\gamma^\nu T^A \psi \,, \tag{4.30}$$

which imply that we can rewrite the derivative operator in terms of fermion fields as

$$\mathcal{O} \sim (\partial X)^2 \sim \left(\psi^2\right)^2 \sim \psi^4 \,, \tag{4.31}$$

namely in the form of four-fermion operators. Therefore we should not include operators of the form $(\partial X)^2$ in our WET Lagrangian, since these are not independent from the four-fermion operators that we already include because they are related by the EoM of the classical Lagrangian (recall also the corresponding discussion in the case of the two-scalar EFT).

Hence at $d = 5$ dimensional counting tells us that we only have one class of operator contributing to the WET Lagrangian, namely $\mathcal{O} \sim \psi^2 X$. One example of this class of operators is the following

$$\mathcal{O}_{uG} = \bar{u}_L \sigma^{\mu\nu} T^A u_R G^A_{\mu\nu} \,, \tag{4.32}$$

where the gluon field strength tensor is given by

$$G^A_{\mu\nu} = \partial_\mu G^A_\nu - \partial_\nu G^A_\mu - g_s f^{ABC} G^B_\mu G^C_\nu \,. \tag{4.33}$$

The $\mathcal{O}_{uG}$ operator hence couples a right-handed quark with a left handed quarks with one or two gluon fields. One clearly different feature of this operator as compared to the $d = 4$ is the presence of chiral interactions, which are absent from the $\mathcal{O}(v^0)$ Lagrangian. These chiral interactions, arising at $\mathcal{O}(v)$, reflect the fact that the UV theory is also chiral. Note that the operator Eq. (4.32) is gauge invariant under only vector gauge interactions, showing that chiral interactions are here not a consequence of the gauge symmetry but rather a general consequence of how the EFT is constructed from the bottom up.

It is useful to reflect on which kind of effective interactions are induced by Eq. (4.32) which are absent from the WET Lagrangian at the zeroth order in the power expansion, which as we have discussed above is nothing but the SM Lagrangian after EWSB restricted to QCD and QED and with only light fields as degrees of freedom. Fig. 4.2 highlights the interactions which are generated by the dimension-5 WET operator of Eq. (4.32): a contact two-quark-two-gluon vertex (top) and a quark-quark-gluon vertex with derivative coupling (bottom). The four-point vertex is a new interaction, which is absent from the leading-power Lagrangian. Instead, the leading power Lagrangian Eq. (4.27) already includes a quark-quark-gluon interaction (this is part of the QCD Lagrangian) but the one in the bottom panel of Fig. 4.2 has both a different structure (it is a chiral interaction, while the QCD one is vectorial) and in terms of strength (it is a derivative coupling whose amplitude grows with energy, while the QCD counterpart is constant).

The example of Fig. 4.2 highlights the main effects of adding the higher-dimensional operators to the leading-power Lagrangian when constructing an EFT from the bottom up:

- Completely new interaction vertices may arise.
- Existing interactions may be modified, by instance by introducing a new dependence with the energy or with the fermion chirality.

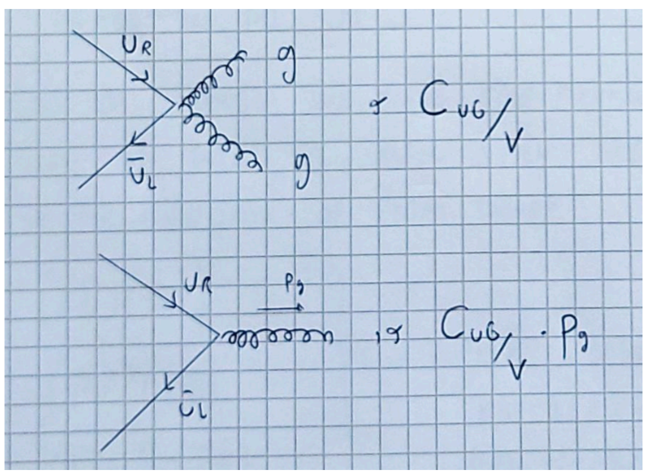

Figure 4.2: Interactions which are generated by the dimension-5 WET operator of Eq. (4.32): a contact two-quark-two-gluon vertex (top) and a quark-quark-gluon vertex with derivative coupling (bottom).

It is instructive to think about which kinds of low-energy processes (here "low-energy" means well below the electroweak scale v) are modified by effective interactions such as those induced by the dimension-5 operator Eq. (4.32). For instance, you can convince yourself that low-energy Drell-Yan production at hadron colliders, $pp \to q\bar{q} + X$, would be modified in the presence of such operators.

Crucially, since we are constructing the WET from the bottom up, and not top-down matching to the SM, our Lagrangian may contain operators that are absent from the SM, generated by some possible new physics well above the electroweak scale. In other words, the WET Lagrangian will contain three classes of higher-dimensional operators:

(a) Operators that are generated by starting from the SM and integrating out the heavy fields. These operators in the WET represent interactions that are already present in the SM, now presented in handier way at low energies since only the light fields are present.

(b) Operators that are absent in the SM, but that are allowed by the field content and symmetries of the WET. These operators may be generated by some new physics above the EW scale, and are described in terms of what is called the Standard Model Effective Field Theory (SMEFT) that we will see later in the course. In this case, for a top-down construction one needs to match the full SMEFT (and not only the SM) to the WET.

(c) A combination of the two: WET operators that receive both a SM contribution and a contribution from possible new particles and forces beyond the SM, such as those parametrized by the SMEFT.

Let's now turn to consider the dimension-6 operators that can be found in the WET Lagrangian. As discussed above, these are operators with structure $\mathcal{O} \sim X^3$ or $\mathcal{O} \sim \psi^4$, hence either purely gauge or four-fermion operators. An example of the purely gauge operator is the all-gluon operator given by:

$$\mathcal{O}_G = f^{ABC} G_\mu^{A\nu} G_\nu^{B\rho} G_\rho^{C\mu} \,, \tag{4.34}$$

which has associated interaction vertices of the form

$$(\partial G)^3 \,, (\partial G)^2 \, G^2, (\partial G) \, G^4, G^6, \tag{4.35}$$

hence contributing to multi-gluon production for example at hadron colliders. This is an example of a WET operator which is absent from the SM[4], and that can only be generated by new physics well above the electroweak scale. In other words, if we match the WET to the SM (at tree level) we would find that the associated Wilson coefficient vanishes, $c_G = 0$. This is of course fine if we assemble the WET from the bottom up, since one is being agnostic about the specific UV completion of the theory, and we don't specify whether it is the SM or something else.

In addition to the purely gauge operators, the WET includes of course the four-fermion operators, some of them we have seen already. These operators can be classified depending on the chirality ($\bar{L}L\bar{R}R$, $\bar{L}L\bar{L}L$, $\bar{R}R\bar{R}R$) and depending on their fermionic field content (purely leptonic, semi-leptonic, and non-leptonic). As for the other operators, the four-fermion operators in the WET may arise from the SM upon integrating out the heavy fields (an example being the $\bar{L}L\bar{L}L$ operators such as Eq. (4.12)), others vanish if matching to the SM, and a combination of the two cases.

An example of a semileptonic operator of the $\bar{L}L\bar{L}L$ category arising in the WET at $d = 6$ would be the following:

$$\mathcal{O}_{\nu u}^{V,LL} = \left(\bar{\psi}_\nu^L \gamma^\mu \psi_\nu^L\right) \left(\bar{\psi}_u^L \gamma_\mu \psi_u^L\right) \,, \tag{4.36}$$

which results into a four-point local interaction between two neutrinos and two quarks. This operator is generated by matching to the SM at tree level by considering by the exchange of a Z boson via the $q\bar{q} \to Z \to \nu\bar{\nu}$ process. Again, although this operator of the WET is generated by integrating out heavy SM fields, its Wilson coefficient $c_{\nu u}^{V,LL}/v^2$ would also receive contributions from new physics beyond the SM, for example due to the exchange of a Z' boson with a mass in the TeV scale. These various effects are summarized in Fig. 4.3: Eq. (4.36) generates a four-fermion interaction (left). This four-fermion interaction arises from SM diagrams such as those of the middle panel, via a Z boson exchange. But the WET wilson coefficient does not need to coincide with the SM one: if we have for example a heavy Z' BSM boson at the

[4] Only in the case of tree-level matching. This operator can be generated in the SM at the loop level, but is then suppressed by powers of the SM couplings.

TeV scale (right panel), upon integrating it out its couplings and masses would also contribute to the Wilson coefficient. In the latter case, one needs to match the WET to the SMEFT, as will be briefly discussed in subsequent lectures.

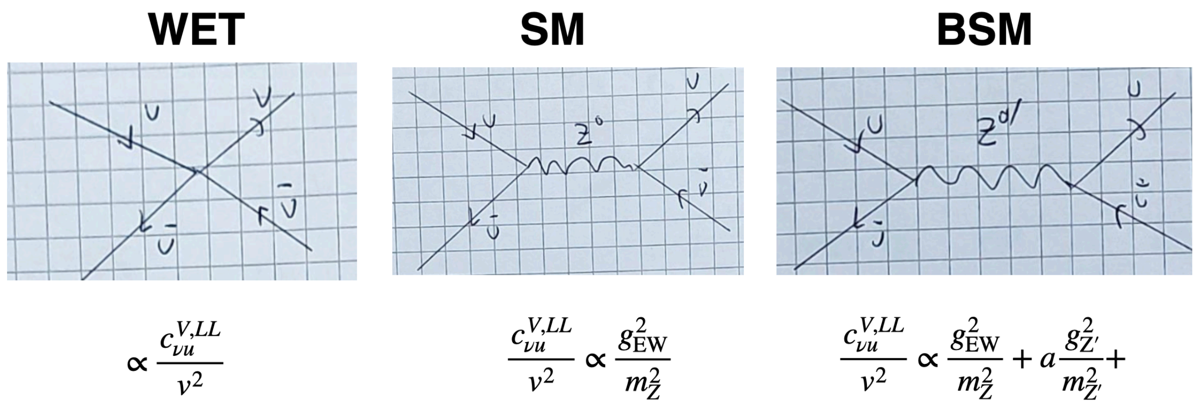

Figure 4.3: The semi-leptonic WET operator Eq. (4.36) generates a four-fermion interaction (left). This four-fermion interaction arises from SM diagrams such as those of the middle panel, via a Z boson exchange. But the WET Wilson coefficient does not need to coincide with the SM one: if we have for example a heavy Z' BSM boson at the TeV scale (right panel), upon integrating it out its couplings and masses would also contribute to the Wilson coefficient.

4.4 Operator mixing in the WET

The WET is particularly useful to compute low-energy processes such as B-meson decays of the type

$$\bar{B}^0 \to D_s^- + \pi^+ \,. \tag{4.37}$$

Fig. 4.4 displays Feynman diagrams representing the B-meson decay $\bar{B}^0 \to D_s^- + \pi^+$ in the WET (left) and in the SM (right panel). Since the mass of the bottom quark is much smaller that the mass of the W boson, $m_b \ll m_W$, it is indeed justified to describe this process in the WET where the W boson has been integrated out.

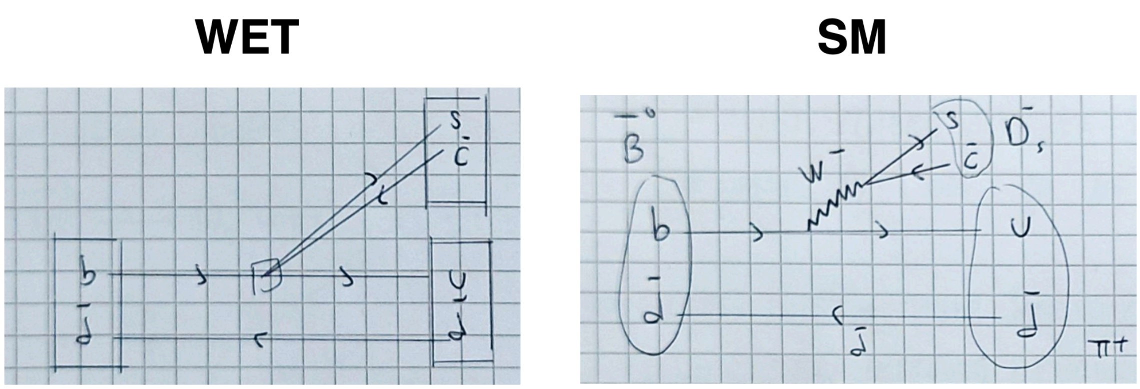

Figure 4.4: Feynman diagrams representing the B-meson decay $\bar{B}^0 \to D_s^- + \pi^+$ in the WET (left) and in the SM (right panel).

Dimensional analysis tells us that the amplitude for this decay will be $\mathcal{A} \propto m_b^2/m_W^2$. One could argue that higher order terms in the EFT expansion are power-suppressed and can be neglected. However, if one computes loop corrections to this process in the SM, one finds corrections which have a prefactor of the form

$$\alpha_s \ln \frac{m_W^2}{m_b^2} \sim 0.2 \times \ln(250) \sim \mathcal{O}(1)\,, \tag{4.38}$$

hence this loop correction, while formally subleading, is enhanced by large logs in the ratio m_W/m_b which then threaten the perturbative convergence of the calculation. In other words, our effective expansion parameter has become non-perturbative.

It should be clear that large logarithms of the form of Eq. (4.38) cannot be generated by calculations carried out in the WET, for the reason that m_W does not appear in its Lagrangian (and the same holds for any other heavy SM particle that may generate large logarithms). In order to account for these effects while keeping the simplicity of the EFT description, we can use the so-called renormalization group improved perturbation theory. Let's illustrate its main idea using the $\bar{B}^0 \to D_s^- + \pi^+$ decay considered above. In the WET, the relevant operator describing this B-meson decay is the following:

$$\mathcal{L}_{\rm WET} \supset \frac{c}{\Lambda^2}\left(\bar{s}_L\gamma^\mu c_L\right)\left(\bar{u}_L\gamma^\mu b_L\right)\,, \tag{4.39}$$

which is hence of the $(\bar{L}L)(\bar{L}L)$ category. By means of tree-loop matching to the SM, it is easy to determine the value of the associated Wilson coefficient

$$\frac{c}{\Lambda^2} = -\frac{4G_F}{\sqrt{2}} V_{cs}^* V_{ub}\,, \tag{4.40}$$

and this coefficient may be different in case of BSM effects. Actually, once we take into account color indices, the operator in Eq. (4.41) is not the only one contributing to this specific B meson decay. One has

$$\mathcal{L}_{\rm WET} \supset L_1\mathcal{O}_1 + L_2\mathcal{O}_2 \tag{4.41}$$

with

$$\mathcal{O}_1 = \left(\bar{s}_L\gamma^\mu b_L\right)\left(\bar{u}_L\gamma^\mu c_L\right)\,, \qquad L_1 = -\frac{1}{N_c}\frac{4G_F}{\sqrt{2}} V_{cs}^* V_{ub} \tag{4.42}$$

$$\mathcal{O}_2 = \left(\bar{s}_L\gamma^\mu T^A b_L\right)\left(\bar{u}_L\gamma^\mu T^A c_L\right)\,, \qquad L_2 = -\frac{8G_F}{\sqrt{2}} V_{cs}^* V_{ub}\,, \tag{4.43}$$

which are obtained after Fierz relations (see the next section). However, none of these two operators will generate the large logarithm that arises in the SM calculation in the presence of QCD loop corrections. The reason is that we carry out the matching between the SM and the WET at $\mu = m_W$, where there are no large logs. Instead, one should match the WET and SM calculations at $\mu \sim m_b$, which this way generates the large-logs correctly in the WET amplitudes. Indeed, this is an intrinsic flexibility of the EFTs: we can match with the UV theory at any scale μ provided we are in a kinematic region where both the EFT and the UV theory are simultaneously applicable.

One can show that if we match the WET and the SM at $\mu = m_b$ we not only reproduce the large logarithm present in the one-loop calculation of the SM, but actually resums the complete tower of such logarithms to all orders in perturbation theory. For this, we note that the scale dependence of the Wilson coefficients L_i with μ is absent at tree level but instead arises at the level of one-loop calculations. Once we compute one-loop corrections to the WET amplitudes, we find the following relation between the relevant

Wilson coefficients

$$\frac{d}{d\ln\tilde{\mu}}\vec{L} = \frac{\alpha_s}{4\pi}\begin{pmatrix} 0 & 6C_F/N_c \\ 12 & -12/N_c \end{pmatrix}\vec{L}\,, \tag{4.44}$$

whose solution *mixes* the two Wilson coefficients L_1 and L_2 as the matching scale μ is varied. For example, assume that at some reference matching scale μ_0 we have $L_1 = 0$ and $L_2 = A$. One can then solve the RGE equations to get

$$\frac{d}{d\ln\tilde{\mu}}L_1 = \frac{6C_F}{N_c}L_2\,, \tag{4.45}$$

which shows that, due to *operator running and mixing* even if $L_1(\mu = \mu_0) = 0$, we will have that all other scales in general one finds $L_1(\mu \neq \mu_0) \neq 0$. From the practical point of view, it is best to solve the RGEs in the basis that makes these equations diagonal. The general solutions for this case are

$$L^{\pm}(\tilde{\mu}) = L^{\pm}(\tilde{\mu}_0 = m_W)\exp\left(\int_{\alpha_s(m_W)}^{\alpha_s(\tilde{\mu})} d\alpha \frac{\alpha}{4\pi}\frac{\gamma^{\pm}(\alpha)}{\beta(\alpha)}\right) \tag{4.46}$$

in terms of the QCD beta function, and where the eigenvectors of the anomalous dimension matrix are given by

$$\gamma^{\pm} = \gamma\left(\pm 1 - \frac{1}{N_c}\right) \tag{4.47}$$

Crucially, by solving the RGEs we generate the large logs in m_b/m_W that are generated by the NLO calculation in the SM and that are absent from the WET (without the RGE improvement). Actually, these logarithms are resummed to all orders in perturbation theory, which is a significant improvement as compared to fixed-order calculations. To show this explicitely, we solve the equation to come with the solution

$$L^{\pm}(\tilde{\mu}) = L^{\pm}(m_W)\left(\frac{\alpha_s(m_W)}{\alpha_s(\tilde{\mu})}\right)^{\gamma^{\pm}/2\beta_0} \tag{4.48}$$

which by using the solution of the RGE for the strong coupling constant

$$\frac{\alpha_s(\tilde{\mu})}{\alpha_s(m_W)} = \left(1 + \frac{\alpha_s(m_W)}{4\pi}\ln\left(\frac{\mu^2}{m_W^2}\right)\right)^{-1}\,, \tag{4.49}$$

results into the following solution for the running and mixing of the WET Wilson coefficients

$$L^{\pm}(m_b) = L^{\pm}(m_W)\exp\left(\frac{\gamma^{\pm}\alpha_s(m_W)}{8\pi\beta_0}\ln\frac{m_b^2}{m_W^2}\right) \tag{4.50}$$

which shows the large logarithms that we wanted to reproduce and to resum. Expanding the exponential in particular reproduces the logarithm of the fixed order calculation that we were worried with to begin with.

In summary, what we have learned concerning operator running and mixing:

- Tree-level matching relations at $\mu = m_W$ do not generate large logarithms.
- There large logarithms appear in the UV theory, so they should be somehow accounted for in the EFT.
- Using the renormalisation group equations we can resum to all orders these potentially large logarithms in the EFT to obtain more accurate predictions.

UNIVERSITEIT VAN AMSTERDAM

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Effective Field Theory

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5 Lecture 5: Weak EFT part 2 + SM primer

5.1 Brief recap

Covered in the previous lecture:

- At energies much below the electroweak scale, we can integrate out the W, Z (and t, h).
- E.g. $\mu \to \nu_\mu e \bar{\nu}_e$ can be described by a four-fermion operator by expanding the W propagator, corresponding to the power counting $\delta \sim E/M_W$.
- By measuring the μ decay rate and assuming the coefficient of this operator to be c/Λ^2 with $c = \mathcal{O}(1)$, we conclude that the scale of "new" physics is $\Lambda \sim 300$ GeV, which is of the same order as the electroweak scale.
- In constructing the LEFT, we use the $SU(3)_c \times U(1)_{\text{em}}$ gauge symmetries.
- The Standard Model (SM) also has accidental symmetries, such as a global $U(1)$ symmetry corresponding to baryon number conservation. This will be inherited by the LEFT if we match to the SM, but should *not* be assumed when searching for BSM physics.

Certain models of BSM physics predict enhanced rates for rare hadronic decays of e.g. B mesons, which is the focus of LHCb. These low energy processes are described by LEFT. There had been a lot of excited about the violation of lepton-flavor universality in $B \to K\ell^+\ell^-$ decays, with a $> 3\sigma$ deviation from the Standard Model. However, in their update last November, LHCb found results consist within 1σ of the Standard Model and they seem to have previously underestimated one of their systematic effects. (Belle didn't see this deviation.)

5.2 Outline for today

- Renormalization group and resummation.
- Including spin and internal symmetries when constructing operator bases: Fierz.
- Standard Model primer (if time).

5.3 Renormalization group and resummation

If one e.g. calculates perturbative corrections to B-meson decays in the standard model, one obtains terms of the form

$$\Gamma = \Gamma_0 \Big(1 + \sum_{1 \le m \le n} c_{n,m} \alpha_s^n \ln^m (M_W^2/m_b^2) + \dots \Big)\,, \tag{5.1}$$

where Λ_0 is the result at leading order. Even though these terms are formally suppressed by powers of $\alpha_s \sim 0.1$ this is largely compensated for by the large $\ln(M_W^2/m_b^2) \approx 6$.

LEFT allows you to resum these logarithms, which amounts to including the leading logarithms (terms with $m = n$), next-to-leading logarithms ($m = n - 1$), etc in eq. (5.1), depending on the desired level of accuracy. Let me first sketch how this works: In LEFT, only the Wilson coefficient L depends on M_W, while m_b only enters in the matrix element of the operator $\langle O \rangle$:

$$1 + c_{1,1}\alpha_s \ln(M_W^2/m_b^2) + \cdots = \underbrace{\big(1 + c_{1,1}\alpha_s \ln(M_W^2/\mu^2)\big)}_{L} \underbrace{\big(1 + c_{1,1}\alpha_s \ln(\mu^2/m_b^2)\big)}_{\langle O \rangle} + \dots . \tag{5.2}$$

Now L and $\langle O \rangle$ won't contain large logarithms if evaluated at $\mu \sim M_W$ and $\mu \sim m_b$, respectively. In the end we need them at the same scale, for which we use the renormalization group

$$\frac{\mathrm{d}L}{\mathrm{d}\ln\mu} = -2c_{1,1}L\,, \qquad \frac{\mathrm{d}\langle O \rangle}{\mathrm{d}\ln\mu} = 2c_{1,1}\langle O \rangle\,. \tag{5.3}$$

Solving this differential equation

$$L(\mu = m_b) = \exp\big[c_{1,1}\alpha_s \ln(M_W^2/m_b^2)\big] L(\mu = m_W)\,, \tag{5.4}$$

so the large logarithms exponentiate. As we will now discuss the reality is more complicated: there is mixing between operators. Furthermore, I ignored that α_s itself depends on μ.

Let's make this more concrete for $\bar{B}^0 \to D_s^- \pi^+$, which is based on $b \to u\bar{c}s$ (**Draw)**). Consider

$$\begin{aligned}
\mathcal{L}_{\mathrm{EFT}} &= L_1 O_1 + L_2 O_2 + \dots \\
O_1 &= (\bar{s}_L \gamma^\mu b_L)(\bar{u}_L \gamma_\mu c_L)\,, \\
O_2 &= (\bar{s}_L \gamma^\mu T^A b_L)(\bar{u}_L \gamma_\mu T^A c_L)\,.
\end{aligned} \tag{5.5}$$

Relating the bare quantities (with tilde) to the renormalized ones (without tilde)

$$\tilde{q} = Z_q^{1/2} q\,, \qquad \tilde{L}_i = Z_{ij} L_j\,, \tag{5.6}$$

you should have obtained during last Friday's exercises that

$$Z_q = 1 - \frac{\alpha_s C_F}{4\pi}\frac{1}{\epsilon}\,, \qquad Z_{ij} = \delta_{ij} + \frac{\alpha_s C_F}{4\pi}\frac{1}{\epsilon}\begin{pmatrix} 0 & \frac{3C_F}{N_c} \\ 6 & -\frac{6}{N_c} \end{pmatrix}_{ij}. \tag{5.7}$$

We can now obtain the evolution for L_i, by using that the bare $\tilde{L}_i$ is μ-independent:

$$0 = \frac{\mathrm{d}\tilde{L}_i}{\mathrm{d}\ln\mu} = \frac{\mathrm{d}Z_{ij}}{\mathrm{d}\ln\mu} L_j + Z_{ij}\frac{\mathrm{d}L_j}{\mathrm{d}\ln\mu} \quad \to \quad \frac{\mathrm{d}L_i}{\mathrm{d}\ln\mu} = \underbrace{-Z_{ij}^{-1}\frac{\mathrm{d}Z_{jk}}{\mathrm{d}\ln\mu}}_{\gamma_{ik}} L_k\,. \tag{5.8}$$

The anomalous dimension γ is at order α_s given by

$$\gamma = \frac{\alpha_s C_F}{2\pi}\begin{pmatrix} 0 & \frac{3C_F}{N_c} \\ 6 & -\frac{6}{N_c} \end{pmatrix}, \tag{5.9}$$

using that

$$\frac{\mathrm{d}\alpha_s}{\mathrm{d}\ln\mu} = -2\epsilon\,\alpha_s + \mathcal{O}(\alpha_s^2)\,. \tag{5.10}$$

It is straightforward to solve this differential equation numerically. To do it analytically with a running coupling, the following trick is convenient

$$\frac{\mathrm{d}\alpha_s}{\mathrm{d}\ln\mu} = \beta(\alpha_s) \quad \to \quad \mathrm{d}\ln\mu = \frac{\mathrm{d}\alpha_s}{\beta(\alpha_s)}\,, \tag{5.11}$$

where $\beta(\alpha_s) = -\beta_0\alpha_s^2/(2\pi) + \mathcal{O}(\alpha_s^3)$ is the QCD beta function for $\epsilon = 0$.

If one would measure this decay, one could extract $C(m_b)$, and this discussion of the scale dependence might seem irrelevant. However, if we wanted to compare/combine with measurements at a different experiment we still need to use it. **(Discuss Wilsonian renormalization picture?)**

5.4 Spin and internal symmetries

For the scalar EFT, we have seen that the operator basis can be simplified by using partial integration and equations of motion. With a $\phi \to -\phi$ symmetry, the first derivative operator that can't be eliminated appears at dimension 8: $\ell^2(\partial_\mu\partial_\nu\phi)(\partial^\mu\partial^\nu\phi)$.

Things become more complicated when fields have spin and/or transform under internal symmetries. We know that the Lagrangian should be a Lorentz scalar and should be invariant under internal symmetries. The challenge is to construct a basis of operators that is complete and linearly independent. Let's start with an example: we can have the four-fermion operators $\bar\psi_1\Gamma\psi_2\,\bar\psi_3\Gamma'\psi_4$, where a complete basis of $\Gamma = \{1, \gamma^\mu, \sigma^{\mu\nu} = \frac{\mathrm{i}}{2}[\gamma^\mu, \gamma^\nu], \mathrm{i}\gamma^\mu\gamma_5, \gamma_5\}$. Indeed, these are $4\times 4 = 16$ matrices and $1 + 4 + 4\times 3/2 + 4 + 1 = 16$. Lorentz invariance requires that the Lorentz indices in Γ and Γ' must be contracted.

Q: How many possibilities are there? Answer: 10

$$\begin{aligned}
&1\otimes 1,\ \gamma_5\otimes\gamma_5,\ 1\otimes\gamma_5,\ \gamma_5\otimes 1,\\
&\gamma^\mu\otimes\gamma_\mu,\ \gamma^\mu\gamma_5\otimes\gamma_\mu\gamma_5,\ \gamma^\mu\otimes\gamma_\mu\gamma_5,\ \gamma^\mu\gamma_5\otimes\gamma_\mu,\\
&\sigma^{\mu\nu}\otimes\sigma_{\mu\nu},\ \epsilon^{\mu\nu\rho\sigma}\sigma_{\mu\nu}\otimes\sigma_{\rho\sigma}\,.
\end{aligned} \tag{5.12}$$

This can be further reduced if discrete symmetries, such as parity, are imposed.

To feel confident that we got everything, we can use group theory: The Lie algebra of the Lorentz group is equivalent to that of $SU(2)\times SU(2)$ and we can label representations by their corresponding (j_1, j_2). E.g. scalar is $(0,0)$ and Dirac fermion is $(\frac{1}{2},0)\oplus(0,\frac{1}{2})$. Now we can count the number of invariants:

$$[(\tfrac{1}{2},0)\oplus(0,\tfrac{1}{2})]^4 = [(1,0)\oplus 2(0,0)\oplus(0,1)\oplus 2(\tfrac{1}{2},\tfrac{1}{2})]^2 = (1+4+1+4)(0,0)+\dots\,, \tag{5.13}$$

using the standard formula for addition of angular momentum to calculate $j_1\otimes j_2$. We indeed find 10.

One could have equally well considered the operator $\bar\psi_1\Gamma\psi_4\,\bar\psi_3\Gamma'\psi_2$. From the counting we did above, we know that we have already obtained all invariants. Indeed, there is a Fierz identity, that allows you to change which fields are contracted with each other:

$$\delta_{ii'}\delta_{j'j} = \tfrac{1}{4}\Gamma^a_{j'i'}\Gamma_{a,ij}\,, \tag{5.14}$$

where there is an implicit sum on a and corresponding Lorentz indices in the Γ's are contracted. Thus one does not need to include this different ordering of fields to have a complete basis. As you will see in the exercises, eq. (5.14) follows from the normalization of Γ^a and completeness:

$$\mathrm{tr}[\Gamma^a \Gamma_b] = 4\delta^a_b\,, \quad X = \tfrac{1}{4}\mathrm{tr}[X\Gamma_a]\Gamma^a\,, \tag{5.15}$$

and then taking $X_{ij} = \delta_{ii'}\delta_{j'j}$. One can then use this to e.g. show

$$\begin{aligned}\bar\psi_1\gamma^\mu P_L\psi_4\,\bar\psi_3\gamma_\mu P_L\psi_2 &= \bar\psi_1\gamma^\mu P_L\psi_2\,\bar\psi_3\gamma_\mu P_L\psi_4\,,\\ \bar\psi_1\gamma^\mu P_L\psi_4\,\bar\psi_3\gamma_\mu P_R\psi_2 &= -2\bar\psi_1 P_R\psi_2\,\bar\psi_3 P_L\psi_4\,,\end{aligned} \tag{5.16}$$

etc. Care must be taken here that the spinors fields are anti-commuting and that this gives extra signs when rearranging the fields.

When the ψ_i are no longer different fields, things become more complicated. For example,

$$\bar\psi\gamma^\mu P_L\psi\,\bar\psi\gamma_\mu P_R\psi = -2\bar\psi P_R\psi\,\bar\psi P_L\psi\,, \tag{5.17}$$

reducing the basis. This case can still be treated using group theory, by looking at the (anti-)symmetric part of the tensor product for bosonic (fermionic) fields. There is some beautiful math (plethystic exponential) that one of my master students used a few years ago to automate the counting of operators for the Standard Model Effective Field Theory.

The story is very similar for internal symmetries. E.g. if ψ_α has 3-components and transforms as $\psi_\alpha \to U_{\alpha\beta}\psi_\beta$ with $U \in SU(3)$, we can construct the following invariants

$$\bar\psi_{1,\alpha}\Gamma\psi_{2,\alpha}\,\bar\psi_{3,\beta}\Gamma'\psi_{4,\beta}\,, \qquad \bar\psi_{1,\alpha}T^A_{\alpha\beta}\Gamma\psi_{2,\beta}\,\bar\psi_{3,\gamma}T^A_{\gamma\delta}\Gamma'\psi_{4,\delta}\,. \tag{5.18}$$

Here we made the indices $\alpha, \beta, \gamma, \delta$ corresponding to the internal symmetry explicit (we have suppressed the spin indices). The T^A with $a = 1, \dots 8$ are the generators of $SU(3)$ and are Hermitian and traceless. This is complete because $\{1, T^A\}$ is a basis of all 3×3 matrices. (You can also check this with group theory: $(\bar 3 \otimes 3)^2 = (1 \oplus 8)^2 = 2 \cdot 1 + \dots$). One could again have changed which fields are contracted with each other, but this does not yield anything new due to the Fierz identity

$$\delta_{\alpha\delta}\delta_{\gamma\beta} = \frac{1}{3}\delta_{\alpha\beta}\delta_{\gamma\delta} + 2T^A_{\alpha\beta}T^A_{\gamma\delta}\,, \tag{5.19}$$

which is an exercise. So we could have also chosen the following basis:

$$\bar\psi_{1,\alpha}\Gamma\psi_{2,\alpha}\,\bar\psi_{3,\beta}\Gamma'\psi_{4,\beta}\,, \qquad \bar\psi_{1,\alpha}\Gamma\psi_{2,\beta}\,\bar\psi_{3,\beta}\Gamma'\psi_{4,\alpha}\,, \tag{5.20}$$

or, after applying a spin Fierz,

$$\bar\psi_{1,\alpha}\Gamma\psi_{2,\alpha}\,\bar\psi_{3,\beta}\Gamma'\psi_{4,\beta}\,, \qquad \bar\psi_{1,\alpha}\Gamma\psi_{4,\alpha}\,\bar\psi_{3,\beta}\Gamma'\psi_{2,\beta}\,. \tag{5.21}$$

6 The Standard Model Effective Field Theory

The Standard Model of particle physics, often simply called the Standard Model (SM), is an extremely successful quantum field theory that describes almost everything we know and observe about the interactions of elementary particles. The Standard Model correctly predicts a large number of complicated differential distributions in high-energy collisions occuring at the Large Hadron Collider (LHC). At the same time it describes many low-energy processes such as beta-decay processes of muons, neutrons, and even atomic nuclei[5]. One historically important property of the Standard Model is that the theory is renormalizable. This was proven by 't Hooft and Veltman in the 70's and they received the 2002 Nobel prize for this achievement. The renormalizability of the Standard Model was crucial at that time because it meant that the theory gives finite predictions at any order in perturbation theory and could be true up to arbitrary high energy (well up to the Planck scale where quantum gravity effects are expected to play a role).

With what we have learned in this class, we might want to take a different perspective. We know nowadays that while the Standard Model is very successful, it should not be too full of itself. It misses important physics! There is one major experimental observation that is completely at odds with the Standard Model: the fact that neutrinos oscillate. This implies that neutrinos are massive particles but in the original Standard Model Lagrangian (see for instance Weinberg's paper 'a model of leptons') the neutrinos are massless. The Standard Model also fails on cosmological fronts: it does not have a Dark Matter candidate nor does it describe how the universe evolved from the Big Bang into a universe with more matter-than-anti-matter. The Standard Model does not describe cosmic inflation. These problems indicate that the SM is not complete and should be extended by a theory that does describe these phenomena.

Historically, such SM extensions were constructed by adding new degrees of freedom to the theory and/or by adding new symmetries or principles (an important example which combines all of this is the minimal supersymmetric SM). These SM extensions are often motivated not just by the observed SM shortcomings but also by more theoretical arguments (for instance the hierarchy problem or the strong CP problem which are not problems in the sense of inconsistenties with data but more unappealing SM features). A different point of view has also been developed. Instead of thinking about the SM as a complete theory of nature, let's just treat it as an EFT that works well only up to some given energy scale which we will denote by $\Lambda \gg M_W$. You can compare this to Fermi's theory of weak interaction which is a good description as long as $E \ll M_W$.

The SM Lagrangian can be constructed by writing down all terms, containing the SM degrees of freedom, with dimension $d \leq 4$ that are invariant under Lorentz symmetry and $SU_c(3) \otimes SU_L(2) \otimes U_Y(1)$ gauge transformations. The demand that $d \leq 4$ ensures the renormalizability of the SM of course. If we let this constraint go, we obtain the Standard Model Effective Field Theory (SM-EFT). In this light, the SM-EFT is very straightforward to write down. It has the same degrees of freedom and the same symmetry principles as the SM, and the difference is that the SM-EFT contains terms with $d > 4$. These terms, however, can have major consequences as we will see in this lecture.

Having specified the symmetries and degrees of freedom of the SM-EFT, we need to discuss the final crucial feature of any EFT. The power counting. This turns out to be rather straightforward as well as observables can be expanded in a power series of E/Λ where E denotes the typical energy scale of the process we are investigating (E can be related to, for example, the center-of-mass energy of a collision experiment but also to a fixed SM scale such as the mass of a particle. In any case we require $E \ll \Lambda$ for the SM-EFT to make sense). On top, we have the usual perturbative expansion in terms of the SM coupling

[5] Because the Standard Model contains QCD, which becomes non-perturbative at low energies, it is not exactly straightforward to compute SM predictions for processes involving hadrons. EFTs can help with that as well, and we will see an important example of that in the next lecture on chiral perturbation theory.

	Q	u_R	d_R	L	e_R	H
Y	$\frac{1}{6}$	$\frac{2}{3}$	$-\frac{1}{3}$	$-\frac{1}{2}$	-1	$\frac{1}{2}$

Table 1: Hypercharge assignment of Standard Model particles. These assignments are derived below.

constants.

6.1 The Standard Model Lagrangian

Let us first discuss the SM Lagrangian itself. If you have never studied this in detail before, some terms might look unfamiliar but don't fret as the main ideas of the SM-EFT can be understood without knowing all subtleties of the SM. Constructing the SM Lagrangian systematically serves a dual purpose: first of all it reminds us of the SM. Second, a systematic construction of all possible terms already shows the general construction of EFT Lagrangians. In the notes in this section we will give more details for which there is no time in the actual lectures.

For simplicity let us begin with a single generation of fermion fields. The SM contains then a left-handed doublet of quarks and a left-handed doublet of leptons

$$Q = \begin{pmatrix} u_L \\ d_L \end{pmatrix}, \qquad L = \begin{pmatrix} \nu_L \\ e_L \end{pmatrix}. \tag{6.1}$$

Here the fields u_L, d_L, ν_L, and e_L describe left-handed Dirac spinors where the subscript L implies $\Psi_L \equiv P_L\Psi \equiv \frac{1}{2}(1-\gamma_5)\Psi$. Similarly a right-handed field is defined as $\Psi_R \equiv P_R\Psi \equiv \frac{1}{2}(1+\gamma_5)\Psi$. The $P_{L,R}$ are projectors because $P_LP_L = P_L$, $P_RP_R = P_R$, $P_LP_R = P_RP_L = 0$, and note that $P_L + P_R = 1$ such that $\Psi = \Psi_L + \Psi_R$.

The left-handed doublets transform in the fundamental representation of $SU(2)_L$: $Q \to U(x)Q$, $L \to U(x)L$, where $U(x)$ is a space-time dependent $SU(2)_L$ matrix which we will write as

$$U(x) = e^{\frac{i}{2}g\alpha^a(x)\tau^a}, \tag{6.2}$$

in terms of the Pauli matrices τ^a. Both doublets are charged under $U_Y(1)$ as well and they transform as $Q \to e^{ig'Y_Q}Q$, $L \to e^{ig'Y_L}L$, where $Y_{L,Q}$ are called hypercharges that can be read from Table 1.

We also introduced right-handed fields u_R, d_R, and e_R which are not charged under $SU(2)_L$ gauge symmetry (they are $SU(2)_L$ singlets) but do transform under $U(1)_Y$ in similar fashion as Q and L but with different hypercharge (see Table 1).

The quark fields u_L, d_L, u_R, d_R are the only fermion fields that are charged under $SU(3)_c$ and they all transform as $\{u_L, d_L, u_R, d_R\} \to W(x)\{u_L, d_L, u_R, d_R\}$ where $W(x)$ is a space-time dependent $SU(3)_c$ matrix. The lepton fields are $SU(3)_c$ singlets.

Finally, the SM contains a doublet of complex scalar fields H which transforms under $SU(2)_L$ as $H \to U(x)H$ similarly as Q and L. It also transforms under $U(1)_Y$ with hypercharge 1/2 but it does not feel $SU(3)_c$.

Let us now write down systematically all terms up to $d \leq 4$ that are consistent with these symmetries.

6.1.1 Dimension 0

In principle we can add a constant term to our Lagrangian $\mathcal{L}_0 = C$. By itself, this term is meaningless and does nothing but once we include gravity this term describes the cosmological constant (which confusingly is often called Λ but we use that symbol for the SM-EFT breakdown scale). By dimensional reasoning $[C] = 4$ and from an EFT point of view we'd thus expect $C \sim \Lambda^4$. This is extremely far from the observed value of the cosmological constant for any value of $\Lambda > m_W$ (and in fact for much smaller values of Λ as well). This mismatch is called the cosmological constant problem and indicates that something fishy is going on here. We will not discuss this further as we will consider the SM-EFT without gravity.

6.2 Dimension 1

The only dimension-1 term that we could write down would be $\mathcal{L}_1 = H$ which is Lorentz invariant. However, it breaks $U(1)_Y$ and $SU(2)_L$ gauge invariance as H is charged under these symmetries. As such there are no dimension-1 terms in the SM Lagrangian.

6.3 Dimension 2

Here we can use two scalar doublets. The only gauge-invariant term is $H^\dagger H$ and we write

$$\mathcal{L}_2 = \mu^2 H^\dagger H \,, \tag{6.3}$$

where $[\mu] = 1$. Again by EFT expectation we'd have $\mu^2 \sim \Lambda^2$ whereas in reality μ^2 is related to the Higgs mass $m_h \simeq 125$ GeV and the Higgs vacuum expectation value $v \simeq 246$ GeV, and thus $\mu^2 \sim (100\,\text{GeV})^2 \ll \Lambda^2$. This mismatch is called the hierarchy problem and from the point of view of EFT, it is essentially a breakdown of the naive scaling of the dimension-2 coupling constant. In Eric's lecture where you matched a theory with a heavy and a light scalar to and EFT with just the light scalar, the hierachy problem appeared explicitly through $m^2 \sim M^2$ (with a loop factor suppression). The hierarchy problem has been extremely influential in the field of Beyond-the-Standard-Model (BSM) model building and was a major motivation for models such as supersymmetry and technicolor. However, the observation of a relatively light Higgs and no non-SM degrees of freedom at the LHC, indicates that the naturalness problem might not be a good guide to construct BSM theories. Of course the verdict is still out and it might very well be that new particles are discovered in upcoming runs of the LHC or its upgrade (the high-luminosity LHC) or in a possible future collider (such as the FCC, the future circular collider).

6.4 Dimension 3

Here we could write down terms with three scalars, two scalars and one derivative, or 1 scalar and two derivatives. However, the first and the last will not be gauge invariant (note that $(2 \otimes 2 \otimes 2) = (2 \oplus 2 \oplus 4)$ and thus you cannot construct an $SU(2)$ singlet by combining three doublets), while a term with a single derivative and 2 scalars will not be Lorentz invariant.

We could combine two fermions in principle but a gauge invariant combination like $\bar{L}L$ vanishes because

$$\bar{\Psi}_L \Psi_L = (P_L\Psi)^\dagger \gamma^0 P_L \Psi = \Psi^\dagger P_L \gamma^0 P_L \Psi = \bar{\Psi} P_R P_L \Psi = 0 \,, \tag{6.4}$$

where we used $P_L^\dagger = P_L$ since γ^5 is hermitian. Similarly a term like $\bar{u}_R u_R$ vanishes. We also cannot simply write down $\bar{e}_L e_R$ as this is not invariant under $SU(2)_L$.

The conclusion is that there is no dimension-3 term in the SM Lagrangian.

6.5 Dimension 4

6.5.1 Kinetic terms for fermions and gauge fields

Here we get the bulk of the SM Lagrangian. First of all we can write down the kinetic terms for the fermion and scalar fields. These take the form

$$\mathcal{L}_{4,a} = \bar{Q} i \not{D}\, Q + \bar{u}_R i \not{D}\, u_R + \bar{d}_R i \not{D}\, d_R + \bar{L} i \not{D}\, L + \bar{e}_R i \not{D}\, e_R \tag{6.5}$$

where $\not{D} = \gamma^\mu D_\mu$ denotes the covariant derivative which is different depending on the object on which it acts. For instance, the right-handed electron field only has a $U(1)_Y$ charge so we'd get

$$D_\mu e_R = (\partial_\mu - i g' Y_e B_\mu)\, e_R\,, \tag{6.6}$$

where B_μ is the $U(1)$ gauge field. More generally, we can write

$$D_\mu = \partial_\mu - i \frac{g_s}{2}\, A^a_\mu \lambda^a - i \frac{g}{2}\, W^i_\mu \tau^i - i g' Y B_\mu, \tag{6.7}$$

where A^a denote the $SU(3)_c$ gluons ($a = \{1, \cdots 8\}$ and λ_i are the Gellmann matrices) and W^i the $SU(2)_L$ gauge bosons ($i = \{1, 2, 3\}$). In this definition of the covariant derivate it is implies that the term with gluons is only present for quarks, the terms with $SU(2)_L$ gauge bosons only present for the left-handed fermion doublets and the scalar doublet, and in the last term Y denotes the hypercharge of the field on which the derivative acts.

Having added the gauge fields, we also have to write down their kinetic terms. They are given by

$$\mathcal{L}_{4,b} = -\frac{1}{4} G^a_{\mu\nu} G^{a\,\mu\nu} - \frac{1}{4} W^i_{\mu\nu} W^{i\,\mu\nu} - \frac{1}{4} B_{\mu\nu} B^{\mu\nu}\,, \tag{6.8}$$

expressed in terms of the field strengths

$$\begin{aligned} G^a_{\mu\nu} &= \partial_\mu A^a_\nu - \partial_\mu A^a_\mu - g_s f^{abc} A^b_\mu A^c_\nu, \\ W^i_{\mu\nu} &= \partial_\mu W^i_\nu - \partial_\nu W^i_\mu - g \varepsilon^{ijk} W^j_\mu W^k_\nu, \qquad B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu\,. \end{aligned} \tag{6.9}$$

If you are not familiar with the construction of the non-Abelian gauge theories, don't worry too much. We will not really need this in what follows below. What is important to notice is that the non-abelian field strengths have self-interaction terms. Note that these terms all involve 2 vector fields and 2 derivatives and thus are indeed dimension 4.

6.5.2 The scalar sector.

Let's now look at the scalar sector. We can write down two terms with just scalars

$$\mathcal{L}_{4,c} = (D_\mu H)(D^\mu H)^\dagger - \lambda (H^\dagger H)^2\,. \tag{6.10}$$

You should think about why a term such as $(H^\dagger D^2 H)$ is not necessary. The dimension-2 μ^2 term in Eq. (6.3) and the dimension-4 λ term above together form the Higgs potential. While the form of the terms are fixed by symmetry considerations, the signs of the parameters μ^2 and λ turn out to be crucial for the theory. If

μ^2 and λ are both positive, then the potential

$$V(H) = -\mu^2 H^\dagger H + \lambda (H^\dagger H)^2 \,, \tag{6.11}$$

has a minimum at the non-zero field value $\langle H^\dagger H\rangle = \mu^2/(2\lambda) \equiv v^2/2$ (The $\langle \cdots \rangle$ implies that we are talking about the minimum). $v \simeq 246$ GeV is called the Higgs vacuum expectation value (vev).

It is customary to pick a vacuum

$$\langle H \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v \end{pmatrix} . \tag{6.12}$$

The minimum breaks $SU(2)_L \otimes U(1)_Y$ down to a remainder $U(1)_{em}$ although this might not be clear at this point. The associated fields are excitations around the minimum. In principle, because H is a complex doublet, this involves four degrees of freedom and we could write

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} \phi_1 + i\phi_2 \\ v + \phi_3 + i\phi_4 \end{pmatrix} , \tag{6.13}$$

in terms of 4 real fields ϕ_i. While there is nothing wrong with this, it turns out to be more convenient to instead parametrize

$$H = \frac{1}{\sqrt{2}} Y(x) \begin{pmatrix} 0 \\ v + h(x) \end{pmatrix} , \tag{6.14}$$

where $Y(x)$ is a general $SU(2)$ matrix which describes 3 degrees of freedom, and $h(x)$ describes the fourth. In these lecture notes we will now use the freedom to pick a specific gauge, the so-called unitarity gauge, which eliminates $Y(x)$. That is, we do a gauge transformation $H \to U(x)H$ and pick $U(x) = Y(x)^{-1}$ such that

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v + h(x) \end{pmatrix} . \tag{6.15}$$

In the unitarity gauge The degrees of freedom that have disappeared from H will induce mass terms for some of the gauge bosons. To see this, consider the Higgs kinetic term for the minimum configuration

$$\langle D_\mu H \rangle = -\frac{iv}{2\sqrt{2}} \begin{pmatrix} gW^1_\mu - igW^2_\mu \\ -gW^3_\mu + g' B_\mu \end{pmatrix} , \tag{6.16}$$

such that

$$\langle (D_\mu H)(D_\mu H)^\dagger \rangle = \frac{v^2}{8} \left[g^2 \left((W^1_\mu)^2 + (W^2_\mu)^2 \right) + \left(gW^3_\mu - g' B_\mu \right)^2 \right] , , \tag{6.17}$$

and these terms describe masses for three linear combination of gauge bosons.

It turns out (this is not obvious right now) that the combinations

$$W^\pm_\mu = (W^1_\mu \mp iW^2_\mu)/\sqrt{2} \,, \tag{6.18}$$

describe the $W^\pm$ charged weak gauge bosons. We would then like that $\left(gW^3_\mu - g' B_\mu\right) \sim Z_\mu$ so that the neutral gauge boson becomes massive, while the photon A_μ remains massless. We can do so by making a rotation in the W^3_μ-B_μ plane through

$$B_\mu = \cos\theta_W \, A_\mu - \sin\theta_W \, Z_\mu \,, \qquad W^3_\mu = \cos\theta_W \, Z_\mu + \sin\theta_W \, A_\mu \,, \tag{6.19}$$

with the inverse

$$A_\mu = \cos\theta_W\, B_\mu + \sin\theta_W\, W^3_\mu\,, \qquad Z_\mu = \cos\theta_W\, W^3_\mu - \sin\theta_W\, B_\mu\,. \tag{6.20}$$

The angle θ_W is called the Weinberg angle (essentially everything is named after Weinberg in this business). I will from now one use $c_W \equiv \cos\theta_W$ and $s_W \equiv \sin\theta_W$ to save some writing. With this rotation we obtain

$$\big(gW^3_\mu - g' B_\mu\big) = A_\mu(g s_W - g' C_W) + Z_\mu(g c_W + g' s_W)\,, \tag{6.21}$$

and since we want A_μ to be massless this implies $g s_W = g' c_W$ and thus $\tan\,\theta_W = g'/g$. With this, we finally obtain

$$\langle (D_\mu H)(D_\mu H)^\dagger\rangle = \frac{v^2 g^2}{8}\left[2W^+_\mu W^{\mu,-} + \frac{1}{c_W^2} Z_\mu Z^\mu\right]\,, \tag{6.22}$$

and we read off $M^\pm_W \equiv M_W = vg/2$ and $M_Z = M_W/c_W \geq M_W$ where the last inequality holds because $c_W \leq 1$. This if of course observed in nature where $M_W \simeq 80.4$ GeV and $M_Z \simeq 91.2$ GeV. This then gives for the Weinberg angle

$$s_W^2 = 1 - \frac{M_W^2}{M_Z^2} \simeq 0.223\,. \tag{6.23}$$

Let us take a quick look at the terms with the Higgs boson (h). These are now easily worked out and lead to

$$\mathcal{L}_{4,h} = \frac{1}{2}(\partial_\mu h)^2 - \mu^2 h^2 - \lambda v h^3 - \frac{\lambda}{4} h^4\,, \tag{6.24}$$

which implies $M_h^2 = 2\mu^2 = 2v^2\lambda$. Since the Higgs mass has been measured $M_h \simeq 125$ GeV and we know $v \simeq 246$ GeV (from the W mass and determining the value of g from gauge-fermion interactions, see the next subsection) this implies $\lambda = M_h^2/(2v^2) \simeq 0.13$. We should test this by measuring the Higgs self interactions in Eq. (6.24) through the h^3 and h^4 terms. This is a major goal for the particle physics community but is rather complicated because it involves di-Higgs production. So far the Higgs self interactions have not been directly measured.

6.5.3 Fermion-Gauge interactions

Now that we have figured out the physical gauge bosons, let us quickly look back at interactions between fermions and gauge bosons. These arise from the gauge-boson part of the covariant derivatives. For instance for the left-handed lepton doublet and the right-handed electron we have the relevant terms

$$\mathcal{L}_{\rm lepton-gauge} = \frac{g}{2}\bar{L}\gamma^\mu\tau^i L\, W^i_\mu + g' Y_L\, \bar{L}\gamma^\mu L B_\mu + g' Y_e\, \bar{e}_R\gamma^\mu e_R B_\mu\,, \tag{6.25}$$

in terms of the hypercharges Y_L and Y_e. Consider now the couplings to photons by using Eq. (6.19). We then see

$$\begin{aligned}
\mathcal{L}_{\rm lepton-photon} &= \frac{g s_W}{2}\bar{L}\gamma^\mu\tau^3 L\, A_\mu + g' c_W Y_L\, \bar{L}\gamma^\mu L A_\mu + g' c_W Y_e\, \bar{e}_R\gamma^\mu e_R A_\mu \\
&= A_\mu\left[\bar{\nu}_L\left(g' c_W Y_L + \frac{g s_W}{2}\right)\gamma^\mu\nu_L + \bar{e}_L\left(g' c_W Y_L - \frac{g s_W}{2}\right)\gamma^\mu e_L + \bar{e}_R\,(g' c_W Y_e)\,\gamma^\mu e_R\right] \\
&= A_\mu(g s_W)\left[\left(Y_L + \frac{1}{2}\right)\bar{\nu}_L\gamma^\mu\nu_L + \left(Y_L - \frac{1}{2}\right)\bar{e}_L\gamma^\mu e_L + Y_e\,\bar{e}_R\gamma^\mu e_R\right]\,,
\end{aligned} \tag{6.26}$$

where we used the relation $gs_W = g'c_W$ we derived above. We see now that the choice $Y_L = -1/2$ ensures that the coupling to neutrinos vanish. The photon coupling to left-handed electrons is then proportional to

$$gs_W \equiv e \tag{6.27}$$

which we use to define the charge. Note that this relation can be used to extract g from the fine-structure constant $\alpha_{em} = e^2/(4\pi) \simeq 1/137$ which gives

$$g = \sqrt{(4\pi)\alpha_{em}}\frac{1}{s_W} \simeq 0.64 \tag{6.28}$$

and then we can compute $v = 2M_W/g \simeq 250$ GeV. The more accurate value of $v \simeq 246$ GeV can be obtained if we include renormalization-group corrections ot the tree-level relations above.

Finally, experimentally we know that electromagnetism conserves parity so we want e_R to have the same electromagnetic charge as e_L. This then forces us to pick $Y_e = -1$ (in agreement of course with Table 1).

This whole thing can be done for quarks as well. In general for a doublet (so with an $SU(2)_L$ charge) you can write

$$\frac{1}{2}\tau^3 + Y = \mathcal{Q}_{\rm em} \tag{6.29}$$

where $\mathcal{Q}_{\rm em}$ is a diagonal matrix of charges (in units of e). For the lepton doublet L we'd have $\mathcal{Q}_{\rm em} = \mathrm{diag}(0, -1)$ while for the quark doublet Q we'd have $\mathcal{Q}_{\rm em} = \mathrm{diag}(2/3, -1/3)$ and this then gives $Y_L = -1/2$ again and $Y_Q = 1/6$. Since we want the h field to have no charge we also obtain $Y_H = 1/2$.

For right-handed fields without a weak charge we simply have

$$Y = \mathcal{Q}_{\rm em}\,, \tag{6.30}$$

and thus $Y_e = -1$, $Y_u = 2/3$, and $Y_d = -1/3$.

Having specified the hypercharges, we can use Eq. (6.25) and the corresponding equations for quarks, to read off the couplings to $W^\pm$ and Z bosons. I will not give the explicit couplings here.

6.5.4 The Yukawa sector

There is one class of dimension-4 interactions we have not considered and these involve 2 fermions and 1 scalar field without derivatives. The idea here is that the combinations $\bar{L}\,H$ and $\bar{Q}\,H$ are $SU(2)_L$-invariant but not $U(1)_Y$- nor Lorentz-invariant. We can fix this by adding a right-handed fermion field. For instance for leptons we would write

$$\mathcal{L}_{\rm Yukawa} = -\bar{L}H\,y_e\,e_R + \text{h.c.} \tag{6.31}$$

where the $SU(2)_L$ indices of $\bar{L}$ and H are contracted, and the spinor indices of the fermions are contracted as well. y_e is a coupling constant, typically called the Yukawa coupling, which is currently just a complex number but once we move to more generations it will become a matrix. If we now expand around the Higgs minimum we obtain

$$\begin{aligned}\mathcal{L}_{\rm Yukawa} &= -\frac{y_e v}{\sqrt{2}}\,\bar{e}_L e_R\left(1+\frac{h}{v}\right) + \text{h.c.}\\ &= -\frac{y_e v}{\sqrt{2}}\,\bar{e}e\left(1+\frac{h}{v}\right)\,,\end{aligned} \tag{6.32}$$

where in the last step I assumed that y_e is real. This is actually not an assumption as, for a single generation of leptons, any phase can be absorbed into the definition of the electron fields. We see now the all-important electron mass term once we identify

$$y_e = \frac{\sqrt{2}m_e}{v} \,. \tag{6.33}$$

We also see that the Higgs boson h couples to fermions with a strength proportional to the fermion mass divided by v. This is the reason why the LHC experimentalists have observed Higgs coupling to heavy fermions (top and bottom quarks, and tau leptons) while there is weaker evidence for couplings to charm quarks and muons. No observations has been made yet of couplings to lighter fermions.

For down quarks the above construction works the same way, but we replace $L \to Q$, $e_R \to d_R$, and $y_e \to y_d$. We then obtain

$$y_d = \frac{\sqrt{2}m_d}{v} \,. \tag{6.34}$$

For up quarks we have to think a bit harder. So far we constructed an $SU(2)_L$ singlet through $\bar{Q}\,H = \bar{Q}_\alpha H_\alpha$ where $\alpha = 1, 2$ are SU(2) indices. You can also construct an invariant through

$$\epsilon^{\alpha\beta}\bar{Q}_\alpha H^*_\beta = \bar{Q} i\tau_2 H^* \,, \tag{6.35}$$

where $\epsilon^{\alpha\beta}$ denotes the 2-dimensional anti-symmetric Levi-Civita tensor. You can check that this is invariant under infinitesmal $SU(2)_L$ transformations as

$$\delta\bar{Q} = -\frac{ig}{2}\bar{Q}(\theta\cdot\tau) \,, \tag{6.36}$$

$$\delta H^* = -\frac{ig}{2}(\theta\cdot\tau)^* H^* \,, \tag{6.37}$$

and thus

$$\delta\left(\bar{Q}i\tau_2 H^*\right) = \frac{g}{2}\theta^a\,\bar{Q}\left[\tau^a\tau^2 + \tau^2(\tau^a)^*\right]H^* = 0 \,, \tag{6.38}$$

where we used $(\tau^a)^* = -\tau^2\tau^a\tau^2$ and $\tau^2\tau^2 = 1$. It is customary to define

$$\tilde{H} = i\tau^2 H^* = \frac{1}{\sqrt{2}}\begin{pmatrix} v+h \\ 0 \end{pmatrix} \,, \tag{6.39}$$

and write the gauge-invariant term (you should check yourself that this is also $SU(3)_c$ and $U(1)_Y$ invariant!)

$$\mathcal{L}_{\text{Yukawa}} = -\bar{Q}\tilde{H}\,y_u\,u_R + \text{h.c.} = -\frac{y_u v}{\sqrt{2}}\,\bar{u}u\left(1 + \frac{h}{v}\right) \,, \tag{6.40}$$

and thus generates the same mass and Higgs coupling as for electrons and down quarks.

Finally, within the Standard Model neutrinos are massless so we do not have to build a neutrino mass term, but it is clear one could do this by adding a ν_R field and repeat the exercise above. We will not do this here, but instead argue in the next section how neutrino masses appear in the SM-EFT.

6.6 Three generations

Finally we need to understand what happens once we move the three generations. Most of the construction above remains valid but we simply give each fermion object a generation index. For the fermion doublets we now consider

$$L^i = \left\{ \begin{pmatrix} \nu_{e\,L} \\ e_L \end{pmatrix}, \begin{pmatrix} \nu_{\mu\,L} \\ \mu_L \end{pmatrix}, \begin{pmatrix} \nu_{\tau\,L} \\ \tau_L \end{pmatrix} \right\}, \qquad Q^i = \left\{ \begin{pmatrix} u_L \\ d_L \end{pmatrix}, \begin{pmatrix} c_L \\ s_L \end{pmatrix}, \begin{pmatrix} t_L \\ b_L \end{pmatrix} \right\}, \tag{6.41}$$

and for the singlets

$$e^i_R = \{e_R,\, \mu_R,\, \tau_R\} \qquad u^i_R = \{u_R,\, c_R,\, t_R\} \qquad d^i_R = \{d_R,\, s_R,\, b_R\}\,. \tag{6.42}$$

Now the fermion kinetic terms we can always define to be diagonal in generation space. That is, to move to three directions we simply replace

$$i\bar{L}\not{D}L \to \sum_{i=1}^{3} i\bar{L}^i \not{D} L^i\,, \tag{6.43}$$

where from now on I will no longer write the sum explicitly but use the Einstein convention. We do the same for the left-handed quark doublet and the right-handed fermions. However, once we have made this choice to keep the kinetic terms diagonal, the Yukawa terms are no longer diagonal. We get instead

$$\mathcal{L}_{\text{Yukawa}} = -\bar{L}^i H\,[y_e]_{ij}\, e^j_R - \bar{Q}^i H\,[y_d]_{ij}\, d^j_R - \bar{Q}^i \tilde{H}\,[y_u]_{ij}\, u^j_R + \text{h.c.}\,, \tag{6.44}$$

where the Yukawa couplings have become general 3×3 complex matrices. Each complex matrix in principle depends on 18 parameters but fortunately we can get rid of many of them.

We can show this by going to a basis of fields where the Yukawa matrices become diagonal. In this basis, the mass matrix becomes diagonal as well and it is therefore called the mass basis. It's not too hard to explicitly move to this basis and let's do it for, say, the down-quark Yukawa matrix. First note that the combination $y_d y_d^\dagger$ is an hermitian matrix and can be diagonlized by a single unitary matrix U_d

$$y_d y_d^\dagger = U_d M_d^2 U_d^\dagger\,, \tag{6.45}$$

where M_d^2 is a diagonal and real matrix. This implies we can write $y_d = U_d M_d K_d^\dagger$ where K_d is another unitary matrix. We can do the same for y_u and y_e off course but with different U, M, and K matrices. The entire Yukawa Lagrangian can then be written as

$$\mathcal{L}_{\text{Yukawa}} = -\frac{v}{\sqrt{2}}\left(1 + \frac{h}{v}\right)\left[\bar{d}_L U_d M_d K_d^\dagger d_R + \bar{u}_L U_u M_u K_u^\dagger u_R + \bar{e}_L U_e M_e K_e^\dagger e_R\right]\,, \tag{6.46}$$

now we simply redefine $\{d_R,\, u_R,\, e_R\} \to \{K_d d_R,\, K_u u_R,\, K_e e_R\}$ and $\{d_L,\, u_L,\, e_L\} \to \{U_d d_L,\, U_u u_L,\, U_e e_L\}$ and the entire Yukawa Lagrangian has become diagonal

$$\mathcal{L}_{\text{Yukawa,mass}} = -\frac{v}{\sqrt{2}}\left(1 + \frac{h}{v}\right)\left[\bar{d}_L M_d d_R + \bar{u}_L M_u u_R + \bar{e}_L M_e e_R\right]\,. \tag{6.47}$$

The matrices $M_{d,u,e}$ are diagonal and contain the real particle masses (up to a factor $\sqrt{2}/v$). We observe that once we made the mass matrices diagonal and real, the interactions with the Higgs are also generational-diagonal (and real). So in the Standard Model we can look for $h \to \bar{b}b$ or $h \to \bar{s}s$ but we should not expect to find $h \to \bar{b}s$.

Now the field redefinitions are not completely without consequence. We have to redefine the fields in the entire Lagrangian. For most terms this does nothing but there is one exception: the charged weak interactions couple $\bar{u}_L$ to d_L (and $\bar{e}_L$ to ν_L) and in those terms the field redefinitions do not drop out. In particular, in the basis of Eq. (6.43) we have terms

$$\mathcal{L}_{W^\pm} = \frac{g}{\sqrt{2}} W_\mu^+ \left[\bar{u}_L^i \gamma^\mu d_L^i + \bar{\nu}_L^i \gamma^\mu e_L^i\right] + \text{h.c.}\,, \tag{6.48}$$

which after the field redefinitions become

$$\mathcal{L}_{W^\pm} = \frac{g}{\sqrt{2}} W_\mu^+ \left[\bar{u}_L^i \left(U_u^\dagger U_d\right)^{ij} \gamma^\mu d_L^j + \bar{\nu}_L^i U_e^{ij} \gamma^\mu e_L^i\right] + \text{h.c.}\,. \tag{6.49}$$

For the leptonic term this does not matter. We simply redefine $\nu_L \to U_e \nu_L$ and the U_e matrix disappears. We can do this because we had not touched the ν_L field so far. For the quarks however we can no longer perform any field redefinitions. We define

$$V_{CKM} = U_u^\dagger U_d\,, \tag{6.50}$$

which is unitary (because U_u and U_d are unitary) and is called the Cabibbo-Kobayashi-Maskawa matrix. It allows for generation-changing charged weak interactions.

A general 3×3 unitary matrix can be described by 9 parameters: 3 angles and 6 phases (you can see that there are 3 angles because if the matrix would be real it would describe rotations in 3 dimensions and thus contain 3 angles). You can also show that 5 phases can be absorbed into the quark fields such that we are left with 3 angles and 1 phase. The standard parametrization is given by

$$\begin{aligned} V_{CKM} &= \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_{23} & \sin\theta_{23} \\ 0 & -\sin\theta_{23} & \cos\theta_{23} \end{pmatrix} \begin{pmatrix} \cos\theta_{13} & 0 & \sin\theta_{23}\, e^{-i\delta} \\ 0 & 1 & 0 \\ -\sin\theta_{13} e^{i\delta} & 0 & \cos\theta_{13} \end{pmatrix} \begin{pmatrix} \cos\theta_{12} & \sin\theta_{12} & 0 \\ -\sin\theta_{12} & \cos\theta_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \end{aligned} \tag{6.51}$$

where the three angles and the phase are all accurately measured from a large set of observables. In general, this is an extremely well tested part of the Standard Model and a plethora of measurements are described by these few parameters in a consistent way. Fortunately there are few tensions (Google first-row CKM unitarity or flavor anomalies to learn more) between experiments and theoretical predictions such that there is hope, no matter how slim, that the Standard Model reign will finally be toppled!

6.7 The SM-EFT framework

We have now exhausted the construction of the Standard Model Lagrangian. The structure of the interactions are remarkable simply, the main complications arise from the spontaneous symmetry breaking and the associated mixing of the $SU(2)_L$ and $U(1)_Y$ gauge bosons, and the fact that we live in a world with three generations. Let us now start with the EFT interpretation: we say that the Lagrangian derived above is simply the renormalizable part of a more general Lagrangian that contains higher-order terms. The Lagrangian can be written as

$$\mathcal{L} = \mathcal{L}_{SM} + \frac{1}{\Lambda} \sum_i C_i^{(5)} O_i^{(5)} + \frac{1}{\Lambda^2} \sum_i C_i^{(6)} O_i^{(6)} + \dots\,, \tag{6.52}$$

where the $C's$ and $O's$ denote, respectively, couplings and operators of higher and higher dimension. The couplings are usually called Wilson coefficients. For this expansion to make sense we require $\Lambda \gg v \sim M_W \sim M_Z \sim m_t$ and, when using it to analyze experiments with an energy E, $\Lambda \gg E$.

Now we cannot just add any operator we want, they have to fulfill some criteria:

(a) First of all, we assume that we have not missed any light degrees of freedom so far so that the effective higher-dimensional operators just consist out of SM fields. This is not necessarily true and there are beyond-the-SM scenarios that involve light new particles (for example axions or sterile neutrinos) that avoided detection not because they are too heavy to be produced but because they couple to weakly to be produced in sufficient amounts to be detected. We will not discuss such scenarios here.

(b) We want the operators to have the crucial symmetries of the SM: $SU(3)_c \otimes SU(2)_L \otimes U(1)_Y$ gauge symmetry and Lorentz symmetry. This greatly restricts the form and number of higher-dimensional operators. We could make stronger assumptions, for instance by requiring that the operators obey additional symmetries for example the conservation of baryon or lepton number, or CP symmetry. Such additional assumptions might be reasonable but inherently imply assumptions on beyond-the-Standard-Model physics which, because we do not know what is out there, can be dangerous.

The great advantage of the SM-EFT approach is that we do not have to specify a specific beyond-the-SM scenario. Instead, we only assume that the scale of new physics is large (which is reasonable as we do not see new degrees of freedom in our high-energy experiments) and then construct the most general EFT operators consistent with the crucial symmetries of the Standard Model. As such, the SM-EFT approach is model independent. In previous examples in this lecture course we mainly used EFT techniques to simplify our calculations, but we in principle new the UV-complete theory and could, with significantly more effort, have done our computations in the full model. This is not true here: we do not know the high-energy theory and we use the EFT approach to parametrize our ignorance in a systematic fashion. That being said, if we would like to, we can study any model of beyond-the-SM physics explicitly by performing a matching calculation to the SM-EFT Lagrangian at the matching scale Λ. This means that once we connect the SM-EFT Wilson coefficients to data (and the data can come very a very large range of experiments) we are actually testing a very large set of possible beyond-the-SM models.

The downside of the SM-EFT approach is that, unfortunately, the number of operators grows very quickly. We will see that there are many operators already appearing at the level of dimension-six operators. Strictly speaking, the EFT approach requires us to consider all operators equally at a given order in the power counting (thou shall not favor one operator over another!) and because of the large number of Wilson coefficients, a practical analysis becomes tricky. This then leads people to make assumptions about the EFT operators (additional symmetries or only operators with third-generation fermions or whatever) but this dilutes the main advantage of the SM-EFT approach: model independence. In any case, the SM-EFT approach to discovering new physics and to stress-test the SM has grown into a large field. Let us get going.

6.8 Our first steps into new territory: Dimension-5 operators

From power counting the largest effects are expected from terms with the least suppression by powers of $1/\Lambda$ so we should begin with dimension-5 operators. We could combine 5 scalars but this is not gauge invariant. Also we cannot use 4 scalars and a derivative because this is not Lorentz invariant. Pure scalar-operators are thus not possible. We could consider 2 fermions and 2 derivatives. But then we'd get $\bar{\Psi}_L D^2 \Psi_L$ which vanishes or $\bar{\Psi}_L D^2 \Psi_R$ which is not gauge invariant. Similarly, combining a field strength with 2 fermions doesn't work. For instance we could try

$$\bar{L}\sigma^{\mu\nu}e_R B_{\mu\nu} \tag{6.53}$$

but this is again not $SU(2)_L$ nor $U(1)_Y$ invariant.

The only remaining option to generate a dim-5 operator would be to combine 2 fermions and 2 scalars. This is not as easy as it sounds but let's try it. Consider the combinations $\bar{L}\tilde{H}$ and $\tilde{H}^\dagger L$ which are both $SU(3)_c$, $SU(2)_L$, and $U(1)_Y$ singlets. So if we could combine these objects into a Lorentz scalar then we'd be in business! Now naively gluing them together does not work because $\bar{L}L = 0$. But it turns out there is a way. This is a bit tricky to see with the four-component spinors that we are using but the combination $L^T CL$ is a Lorentz-scalar, so a possible dimension-5 term is

$$\mathcal{L}_5 = -\frac{C_W^{(5)}}{\Lambda} L_k^T C L_m H_l H_m \epsilon^{kl}\epsilon^{mn} + \text{h.c.} \tag{6.54}$$

This operator is often called, what else, the Weinberg operator. C is the charge conjugation matrix which we will define through $C = -i\gamma_2\gamma_0$ (note that $C^2 = -1$ and $C^\dagger = -C$ so that $C^{-1} = C^\dagger = -C$). It is also useful to write out

$$C = -i\gamma_2\gamma_0 = -i\begin{pmatrix} 0 & \tau_2 \\ -\tau_2 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = -i\begin{pmatrix} \tau_2 & 0 \\ 0 & -\tau_2 \end{pmatrix}. \tag{6.55}$$

In the Higgs minimum we obtain

$$\mathcal{L}_5 = -\frac{C_W^{(5)} v^2}{\Lambda}\nu_L^T C\nu_L\,, \tag{6.56}$$

and the object $[C_W^{(5)} v^2]/\Lambda$ is called the neutrino Majorana mass term. You will practice with this funny-looking term in the exercises, but for now I will just say that this term can correctly describe the observed mass of neutrinos. It comes with some interesting features.

- First of all, the mass scales as v^2/Λ and thus for $\Lambda \gg v$ this somewhat explains why neutrinos are so much lighter than other particles (we know that neutrino masses are of the order of 0.1 eV, a million times smaller than the electron).
- Second, it makes a very clear prediction. The neutrino mass is of the Majorana type which means that neutrinos are Majorana particles (again we work this through in the exercises). What does mean is that neutrinos only have 2 degrees of freedom instead of 4 for usual fermions (like electrons). You can see this from the Lagrangian where both the kinetic and the mass term only depends on $\nu_L = P_L\nu$ and the P_L projector projects out 2 degrees of freedom.
- The Majorana nature can also be seen in a different light. The SM Lagrangian itself has several accidental symmetries[6]. For example if we give all quarks fields (so for all generations) Q, u_R, and d_R the same global phase then the SM Lagrangian does not change. The associated conserved charge is baryon number (B) which ultimately explains why we can classify hadrons into mesons ($B = 0$), hadrons ($B = 1$), and nuclei ($B > 2$). Similarly, the SM is invariant if we give all leptons L and e_R the same global phase and this is associated to the conservation of lepton number. However, the Weinberg operator involves 2 L fields (instead of $\bar{L}$ and L) and is not invariant under a global $U(1)$ phase transformation. This means that the Weinberg operator breaks Lepton number conservation and thus leads to lepton number violating processes.

[6]accidental symmetries are symmetries that we did not put in by hand but instead emerge after constructing the Lagrangian. So in the SM we build a Lagrangian consistent with gauge and Lorentz symmetry but we do not put in, for example, baryon number conservation by hand. This symmetry is an accident and is one of the strong points of the SM as it is a prediction.

All together, SM-EFT comes with a clear prediction. Neutrinos are massive and Majorana states. We have confirmed they are massive (big win for EFT) but have not confirmed they are Majorana (the verdict is still out). New experiments looking for neutrinoless double beta decay (a lepton-number-violating process where two neutrons are transmuted into two protons and two electrons but no neutrinos) are very close to being able to tell whether neutrinos are Dirac or Majorana states. My money is on Majorana of course since that is what SM-EFT predicts.

Let's also be slightly more negative. Imagine that a next-generation experiment confirms neutrinos are Majorana. We could then argue that this is caused by the dimension-5 Weinberg operator. The absolute neutrino mass scale is roughly 0.1 eV so this implies

$$\frac{C_W^{(5)} v^2}{\Lambda} \simeq 0.1\,\mathrm{eV} \qquad \to \qquad \Lambda \simeq \frac{C_W^{(5)} v^2}{0.1\,\mathrm{eV}} \simeq C_W^{(5)}\,(6\cdot 10^{14})\,\mathrm{GeV}\,. \tag{6.57}$$

Now an EFT does not predict the size of the Wilson coefficients but one typically assumes that after taking care of the dimensions through powers of Λ the remainder is not too big or too small. So if we set $C_W^{(5)} \sim 1$, we'd get $\Lambda \sim 10^{15}$ GeV or so. This is a very very very large energy scale. In some sense for the EFT point of view it is good: a larger scale means a better expansion. But for us, it is bad. If this is indeed the scale of beyond-the-SM physics then we will never be able to detect anything beyond neutrinoless double beta decay (a possible exception is proton decay which we will discuss below).

Fortunately, the situation might not be so dire. First of all, there can be several scales of beyond-the-SM physics and it might be that neutrino masses are generated at a very different scale than other operators. Second, you will show in the exercises that a very attractive UV-completion of the Weinberg operator predicts $C_W^{(5)} \sim y_\nu^2$ where y_ν is a neutrino Yukawa coupling. Already in the SM these couplings can be very small (for instance $y_e \sim m_e/v \sim 10^{-5}$). For instance, setting $y_\nu = y_e$ would lower the scale of Λ by 10 orders of magnitude to $\Lambda = 10^4$ GeV which is definitely accessible. What the real scale of lepton number violation is, or whether it is there at all, we unfortunately do not know yet. We have to do some more digging!

6.9 All hell breaks loose: Dimension-6 operators

So far, things went pretty great. We added one operator at dim-5 and it right away solved a huge problem of the SM. Unfortunately, things are going to take a turn for the worse. At the level of dimension-six operators, we can construct a large number of different operators. This is already clear by looking at possible structures that lead to dim-6 operators. We can combine 6 scalars, 4 scalars and 2 derivatives, 2 scalars and 2 field strengths, 2 fermions and 3 scalars, 2 fermions a scalar and a field strength, 2 fermions + 2 scalars + derivative, and, worst of all, we can combine 4 fermions.

By constructing all possible operators and using integration by parts and equations of motion to eliminate redundant operators, Buchmuller and Wyler constructed the full set of dimension-six operators already in 1985. Somewhat embarrisingyly, the field used their basis of operators for many years until a Polish group (Gradzkowski, Iskrzynski, Misiak, Rosiek) identified a few mistakes in the derivation of the basis. They corrected the set of operators and came up with a convenient basis that is complete and independent. This basis is now called the Warsaw basis. The operators are given in Fig. 6.1. These tables are copied from the original literature and the notation is slightly different than used here: φ is used instead of H, the right-handed fermion singlets are denoted by e, u, d instead of e_R, u_R, d_R, and the left-handed doublets are given by q and l instead of Q and L.

You see right away that compared to dimension-4 there are many more structures. Furthermore, for operators with fermions there are many flavor configurations. Take for instance, the operator

$$(H^\dagger H)(\bar{L}_p e_r)H\,. \tag{6.58}$$

This is basically dimension-4 Yukawa interaction (see Eq. (6.31)) multiplied by $H^\dagger H$. This is thus clearly a dimension-six operator which is gauge invariant. The indices p and r denote the generation index and can each take a value $\{1,2,3\}$. So there are actually 3×3 different flavor configuration each with it's own independent (and in this case complex) Wilson coefficient. For four-fermion operators there are even more possible configurations. In total, it turns out there are 2499 independent dimension-six operators. I hope you have no plans for the evening.

6.9.1 Proton decay

It is interesting to look at SM-EFT operators that break accidental symmetries of the SM since these can lead to dramatic effects that cannot happen in the SM. We already saw that dim-5 operators break lepton number conservation, and dim-6 operators can violate baryon number. Consider the operator

$$\mathcal{L}_{\mathrm{B-violation}} = \frac{C^{(6)}_{duue}}{\Lambda^2}\epsilon^{\alpha\beta\gamma}\left[(d_p^\alpha)^T C u_r^\beta\right]\left[(u_s^\gamma)^T C e_t\right]\,, \tag{6.59}$$

where Greek letters indicate $SU(3)_c$ indices. You can see that this operator is clearly $SU(2)_L$ invariant since we only involve right-handed singlets. For right-handed particles the hypercharges are identical to the charges and this operator conserves charge $2\times(2/3)+(-1/3)+(-1)=0$, and you should check yourself that this operator is also $SU(3)_c$ invariant (but $3\otimes 3\otimes 3 = 1+10+10+8$ so 3 quarks can form a singlet and it is completely antisymmetric).

The operator however breaks an accidental symmetry of the SM Lagrangian. If we give each quark field the same overal phase $e^{i\theta_B}$ then the SM stays the same, but this operator clearly does not (it picks up an overall phase factor $e^{3i\theta_B}$ and thus this operators leads to baryon-number violation (the operator also breaks lepton number but we already saw that at the dim-5 level so we are not as impressed). The experimental consequences are dire.

If we set $p=r=s=1$ (so just up and down quarks and an electron) this operator leads to the decay of the proton through

$$p\to\pi^0+e^+\,. \tag{6.60}$$

Now as far as we know, the proton is stable. The Super-Kamiokande experiment (basically a giant water tank) have set the lower limit on the proton life time

$$\tau_p > 1.67\cdot 10^{34}\,\mathrm{y} \tag{6.61}$$

(I always find this absolutely insane considering the lifetime of the universe is only around 10^{10} years). Now it is not completely straightforward to directly match $C^{(6)}_{duu}$ to τ_p directly but you can do a rough estimate which gives

$$\Gamma_p = \tau_p^{-1} = \frac{m_p}{(4\pi)}\frac{\Lambda^4_{\mathrm{QCD}}}{\Lambda^4}|C^{(6)}_{duue}|^2\,, \tag{6.62}$$

where Λ_{QCD} is some hadronic scale and m_p is the proton mass. Let's set $\Lambda_{\mathrm{QCD}}\simeq 300$ MeV and $m_p\simeq 1$

X^3		φ^6 and $\varphi^4 D^2$		$\psi^2\varphi^3$	
Q_G	$f^{ABC} G_\mu^{A\nu} G_\nu^{B\rho} G_\rho^{C\mu}$	Q_φ	$(\varphi^\dagger\varphi)^3$	$Q_{e\varphi}$	$(\varphi^\dagger\varphi)(\bar l_p e_r \varphi)$
$Q_{\widetilde G}$	$f^{ABC} \widetilde G_\mu^{A\nu} G_\nu^{B\rho} G_\rho^{C\mu}$	$Q_{\varphi\Box}$	$(\varphi^\dagger\varphi)\Box(\varphi^\dagger\varphi)$	$Q_{u\varphi}$	$(\varphi^\dagger\varphi)(\bar q_p u_r \widetilde\varphi)$
Q_W	$\varepsilon^{IJK} W_\mu^{I\nu} W_\nu^{J\rho} W_\rho^{K\mu}$	$Q_{\varphi D}$	$\left(\varphi^\dagger D^\mu\varphi\right)^\star \left(\varphi^\dagger D_\mu\varphi\right)$	$Q_{d\varphi}$	$(\varphi^\dagger\varphi)(\bar q_p d_r \varphi)$
$Q_{\widetilde W}$	$\varepsilon^{IJK} \widetilde W_\mu^{I\nu} W_\nu^{J\rho} W_\rho^{K\mu}$				

$X^2\varphi^2$		$\psi^2 X\varphi$		$\psi^2\varphi^2 D$	
$Q_{\varphi G}$	$\varphi^\dagger\varphi\, G_{\mu\nu}^A G^{A\mu\nu}$	Q_{eW}	$(\bar l_p \sigma^{\mu\nu} e_r)\tau^I \varphi W_{\mu\nu}^I$	$Q_{\varphi l}^{(1)}$	$(\varphi^\dagger i \overleftrightarrow{D}_\mu \varphi)(\bar l_p \gamma^\mu l_r)$
$Q_{\varphi\widetilde G}$	$\varphi^\dagger\varphi\, \widetilde G_{\mu\nu}^A G^{A\mu\nu}$	Q_{eB}	$(\bar l_p \sigma^{\mu\nu} e_r)\varphi B_{\mu\nu}$	$Q_{\varphi l}^{(3)}$	$(\varphi^\dagger i \overleftrightarrow{D}^I_\mu \varphi)(\bar l_p \tau^I \gamma^\mu l_r)$
$Q_{\varphi W}$	$\varphi^\dagger\varphi\, W_{\mu\nu}^I W^{I\mu\nu}$	Q_{uG}	$(\bar q_p \sigma^{\mu\nu} T^A u_r)\widetilde\varphi\, G_{\mu\nu}^A$	$Q_{\varphi e}$	$(\varphi^\dagger i \overleftrightarrow{D}_\mu \varphi)(\bar e_p \gamma^\mu e_r)$
$Q_{\varphi\widetilde W}$	$\varphi^\dagger\varphi\, \widetilde W_{\mu\nu}^I W^{I\mu\nu}$	Q_{uW}	$(\bar q_p \sigma^{\mu\nu} u_r)\tau^I \widetilde\varphi\, W_{\mu\nu}^I$	$Q_{\varphi q}^{(1)}$	$(\varphi^\dagger i \overleftrightarrow{D}_\mu \varphi)(\bar q_p \gamma^\mu q_r)$
$Q_{\varphi B}$	$\varphi^\dagger\varphi\, B_{\mu\nu} B^{\mu\nu}$	Q_{uB}	$(\bar q_p \sigma^{\mu\nu} u_r)\widetilde\varphi\, B_{\mu\nu}$	$Q_{\varphi q}^{(3)}$	$(\varphi^\dagger i \overleftrightarrow{D}^I_\mu \varphi)(\bar q_p \tau^I \gamma^\mu q_r)$
$Q_{\varphi\widetilde B}$	$\varphi^\dagger\varphi\, \widetilde B_{\mu\nu} B^{\mu\nu}$	Q_{dG}	$(\bar q_p \sigma^{\mu\nu} T^A d_r)\varphi\, G_{\mu\nu}^A$	$Q_{\varphi u}$	$(\varphi^\dagger i \overleftrightarrow{D}_\mu \varphi)(\bar u_p \gamma^\mu u_r)$
$Q_{\varphi WB}$	$\varphi^\dagger\tau^I\varphi\, W_{\mu\nu}^I B^{\mu\nu}$	Q_{dW}	$(\bar q_p \sigma^{\mu\nu} d_r)\tau^I \varphi\, W_{\mu\nu}^I$	$Q_{\varphi d}$	$(\varphi^\dagger i \overleftrightarrow{D}_\mu \varphi)(\bar d_p \gamma^\mu d_r)$
$Q_{\varphi\widetilde WB}$	$\varphi^\dagger\tau^I\varphi\, \widetilde W_{\mu\nu}^I B^{\mu\nu}$	Q_{dB}	$(\bar q_p \sigma^{\mu\nu} d_r)\varphi\, B_{\mu\nu}$	$Q_{\varphi ud}$	$i(\widetilde\varphi^\dagger D_\mu\varphi)(\bar u_p \gamma^\mu d_r)$

$(\bar LL)(\bar LL)$		$(\bar RR)(\bar RR)$		$(\bar LL)(\bar RR)$	
Q_{ll}	$(\bar l_p \gamma_\mu l_r)(\bar l_s \gamma^\mu l_t)$	Q_{ee}	$(\bar e_p \gamma_\mu e_r)(\bar e_s \gamma^\mu e_t)$	Q_{le}	$(\bar l_p \gamma_\mu l_r)(\bar e_s \gamma^\mu e_t)$
$Q_{qq}^{(1)}$	$(\bar q_p \gamma_\mu q_r)(\bar q_s \gamma^\mu q_t)$	Q_{uu}	$(\bar u_p \gamma_\mu u_r)(\bar u_s \gamma^\mu u_t)$	Q_{lu}	$(\bar l_p \gamma_\mu l_r)(\bar u_s \gamma^\mu u_t)$
$Q_{qq}^{(3)}$	$(\bar q_p \gamma_\mu \tau^I q_r)(\bar q_s \gamma^\mu \tau^I q_t)$	Q_{dd}	$(\bar d_p \gamma_\mu d_r)(\bar d_s \gamma^\mu d_t)$	Q_{ld}	$(\bar l_p \gamma_\mu l_r)(\bar d_s \gamma^\mu d_t)$
$Q_{lq}^{(1)}$	$(\bar l_p \gamma_\mu l_r)(\bar q_s \gamma^\mu q_t)$	Q_{eu}	$(\bar e_p \gamma_\mu e_r)(\bar u_s \gamma^\mu u_t)$	Q_{qe}	$(\bar q_p \gamma_\mu q_r)(\bar e_s \gamma^\mu e_t)$
$Q_{lq}^{(3)}$	$(\bar l_p \gamma_\mu \tau^I l_r)(\bar q_s \gamma^\mu \tau^I q_t)$	Q_{ed}	$(\bar e_p \gamma_\mu e_r)(\bar d_s \gamma^\mu d_t)$	$Q_{qu}^{(1)}$	$(\bar q_p \gamma_\mu q_r)(\bar u_s \gamma^\mu u_t)$
		$Q_{ud}^{(1)}$	$(\bar u_p \gamma_\mu u_r)(\bar d_s \gamma^\mu d_t)$	$Q_{qu}^{(8)}$	$(\bar q_p \gamma_\mu T^A q_r)(\bar u_s \gamma^\mu T^A u_t)$
		$Q_{ud}^{(8)}$	$(\bar u_p \gamma_\mu T^A u_r)(\bar d_s \gamma^\mu T^A d_t)$	$Q_{qd}^{(1)}$	$(\bar q_p \gamma_\mu q_r)(\bar d_s \gamma^\mu d_t)$
				$Q_{qd}^{(8)}$	$(\bar q_p \gamma_\mu T^A q_r)(\bar d_s \gamma^\mu T^A d_t)$

$(\bar LR)(\bar RL)$ and $(\bar LR)(\bar LR)$		B-violating	
Q_{ledq}	$(\bar l_p^j e_r)(\bar d_s q_t^j)$	Q_{duq}	$\varepsilon^{\alpha\beta\gamma}\varepsilon_{jk}\left[(d_p^\alpha)^T C u_r^\beta\right]\left[(q_s^{\gamma j})^T C l_t^k\right]$
$Q_{quqd}^{(1)}$	$(\bar q_p^j u_r)\varepsilon_{jk}(\bar q_s^k d_t)$	Q_{qqu}	$\varepsilon^{\alpha\beta\gamma}\varepsilon_{jk}\left[(q_p^{\alpha j})^T C q_r^{\beta k}\right]\left[(u_s^\gamma)^T C e_t\right]$
$Q_{quqd}^{(8)}$	$(\bar q_p^j T^A u_r)\varepsilon_{jk}(\bar q_s^k T^A d_t)$	Q_{qqq}	$\varepsilon^{\alpha\beta\gamma}\varepsilon_{jn}\varepsilon_{km}\left[(q_p^{\alpha j})^T C q_r^{\beta k}\right]\left[(q_s^{\gamma m})^T C l_t^n\right]$
$Q_{lequ}^{(1)}$	$(\bar l_p^j e_r)\varepsilon_{jk}(\bar q_s^k u_t)$	Q_{duu}	$\varepsilon^{\alpha\beta\gamma}\left[(d_p^\alpha)^T C u_r^\beta\right]\left[(u_s^\gamma)^T C e_t\right]$
$Q_{lequ}^{(3)}$	$(\bar l_p^j \sigma_{\mu\nu} e_r)\varepsilon_{jk}(\bar q_s^k \sigma^{\mu\nu} u_t)$		

Figure 6.1: Dimension-six operators in the Warsaw basis.

GeV, we then obtain

$$\frac{\Lambda}{\sqrt{|C^{(6)}_{duue}|}} > \left(\frac{m_p \Lambda^4_{\rm QCD}}{(4\pi)}\tau_p\right)^{1/4} \simeq 5\cdot 10^{15}\,{\rm GeV}\,. \tag{6.63}$$

So proton decay tests extremely high energy scales! This is essentially what has ruled out the most promising models of grand unification. In these models the SM gauge groups unify to single group (for example $SU(5)$) around 10^{14} GeV. However, in these models quarks and leptons appear in the same multiplets and, after integrating out heavy gauge bosons, the B-violating dimension-six operators are generated. The non-observation of proton decay then rules out these models. The verdict is, again, still not out. Next-generation experiments aim to improve the limit on the proton lifetime by another order of magnitude or two further increasing the limits on Λ.

6.9.2 Modifying the Higgs sector

In the Warsaw basis there are three operators that only contain Higgs fields

$$\mathcal{L}_{\rm Higgs} = \frac{C^{(6)}_H}{\Lambda^2}\,(H^\dagger H)^3 + \frac{C^{(6)}_{H\Box}}{\Lambda^2}(H^\dagger H)\Box(H^\dagger H) + \frac{C^{(6)}_{HD}}{\Lambda^2}(H^\dagger D^\mu H)^*(H^\dagger D^\mu H)\,. \tag{6.64}$$

We are going to study these operators in more detail in the exercises and here we focus on $C^{(6)}_H$ and $C^{(6)}_{HD}$. The first operator modifies the Higgs potential and you will show in the exercises that it shifts the minimum of the Higgs field to

$$\langle H^\dagger H\rangle \equiv \frac{v_T^2}{2}\,,\qquad v_T = v\left(1+\frac{3C^{(6)}_H v^2}{8\lambda\Lambda^2}\right)\,. \tag{6.65}$$

In addition, the $C^{(6)}_H$ operator leads to modified multi-Higgs interactions (also between 5 and 6 h fields) but these are very hard to measure. Because, the $W^\pm$ and Z boson masses depend on v, the $C^{(6)}_H$ operator also shift their masses, even though no gauge fields appear in the operator. Considering also the $C^{(6)}_{HD}$ operator, you will show that

$$\bar{M}_W^2 = \frac{g^2 v_T^2}{4} = M_W^2\left(1+\frac{3C^{(6)}_H v^2}{4\lambda\Lambda^2}\right)\,,\qquad \bar{M}_Z^2 = M_Z^2\left(1+\frac{3C^{(6)}_H v^2}{4\lambda\Lambda^2}+\frac{v^2 C^{(6)}_{HD}}{2\Lambda^2}\right)\,, \tag{6.66}$$

where $\bar{M}_W^2$ and $\bar{M}_Z^2$ indicate the masses in SM-EFT whereas the quantities without bar indicate the SM values.

Now the Z boson mass was measured accurately at the old LEP collider $M_Z^{\rm exp} = 91.1876 \pm 0.0021$ GeV, but this is not immediately helpful. Because we use these measurements to extract the SM parameters. We have to think a little bit harder. Nowadays, the SM parameters of relevance in the electroweak sector (g' g v) are measured from three precisely measured observables:

- Muon decay $\mu^- \to e^- + \bar\nu_e + \nu_\mu$ can be used to extract the Fermi constant $\sqrt{2}G_F = g^2/(8M_W^2) = 1/v^2$. Here we used SM relations.
- The fine-structure constant $\alpha_{\rm em}$ at low energies can be accurately extracted from measurements of the electron magnetic moment. Remember from the SM relations

$$\alpha_{\rm em} = \frac{e^2}{4\pi} = \frac{1}{4\pi}g^2 s_W^2 = \frac{1}{4\pi}\frac{g^2 g'^2}{g^2+g'^2} \tag{6.67}$$

- The mass of the Z-boson which is given by

$$M_Z^2 = \frac{(g^2 + g'^2)v^2}{4}\,. \tag{6.68}$$

From these three precision measurements we can extract the SM values of g, g', and v, and we can then predict other things.

For instance, with some algebra you can show that

$$s_W^2 = \frac{1}{2} - \frac{1}{2}\sqrt{1 - \frac{4\pi\alpha_{\text{em}}}{\sqrt{2}G_F M_Z^2}}\,, \tag{6.69}$$

where the right-hand side only contains measured quantities. Then we can predict the mass of the W boson through the SM relation

$$M_W^2 = M_Z^2(1 - s_W^2) = M_Z^2\left(\frac{1}{2} + \frac{1}{2}\sqrt{1 - \frac{4\pi\alpha_{\text{em}}}{\sqrt{2}G_F M_Z^2}}\right)\,, \tag{6.70}$$

which then predicts $M_W = (80.4335 \pm 0.0094)$ GeV. Now about a year ago there was a new measurement of the W-boson mass by the CDF collaboration. And they found $M_W = 80.361 \pm 0.007$ GeV. This is about 7 standard deviations away from the SM prediction! Now there are many caveats with the CDF measurement as it is inconsistent with other measurements, but let's, for now, take it for granted. We can see if we can explain this by the SM-EFT operators.

What we have to do now is to realize that when we measure G_F, M_Z^2, and α_{em} they in principle already include contributions from potential dimension-six operators. So when we write

$$\bar{M}_W^2 = M_W^2\left(1 + \frac{3C_H^{(6)}v^2}{4\lambda\Lambda^2}\right)\,, \tag{6.71}$$

and we want to use Eq. (6.70) for the value of M_W^2 we have to subtract the dimension-six corrections to the measured values in Eq. (6.70). If you do this consistently and discard $\mathcal{O}(\Lambda^{-4})$ corrections (Mathematica is your friend here), you obtain the relation

$$\frac{\delta M_W^2}{M_W^2} \equiv \frac{\bar{M}_W^2 - M_W^2}{M_W^2} = \frac{v^2 c_W^2}{2(s_W^2 - c_W^2)}\frac{C_{HD}^{(6)}}{\Lambda^2}. \tag{6.72}$$

This then implies that the so-called W-boson anomaly can be explained by

$$\frac{C_{HD}^{(6)}}{\Lambda^2} \simeq \left(\frac{1}{5\,\text{TeV}}\right)^2\,, \tag{6.73}$$

and implies beyond-the-Standard Model physics right around the corner. My feeling is that most people in the field think that the uncertainty on the CDF measurement is underestimated and this is causing the tension with the SM predictions. Otherwise it is hard to understand why ATLAS (one of the LHC detectors) has also measured M_W and in pretty good agreement with SM predictions. In any case, this is a good example how the SM-EFT can be used to understand possible deviations of SM predictions. If an anomaly is confirmed, the next step is to think about what kinds of UV-complete models can induce, at lower energies, corrections to, in this case, $C_{HD}^{(6)}$.

6.9.3 Dipole operators

As a final example, let's study leptonic dipole operators. We study the operator

$$\mathcal{L}_{\text{dipole}} = \frac{1}{\Lambda^2} \left[C^{(6)}_{eB}\right]_{pr} \bar{L}_p \sigma^{\mu\nu} e_r \, H \, B_{\mu\nu} + \text{h.c.} \tag{6.74}$$

where we have kept the generation indices p and r explicit. These operators are called lepton dipole operators because they modify the magnetic and electric dipole moments of leptons. There are several very interesting operators actually and we can consider them one by one.

Lepton flavor violation. If we take $p = 2$ and $r = 1$ and write $B_{\mu\nu} = c_W F_{\mu\nu} - s_W Z_{\mu\nu}$ we get a contribution to the operator

$$\mathcal{L}_{\text{LFV}} = \frac{c_W v}{\Lambda^2\sqrt{2}} \left[C^{(6)}_{eB}\right]_{21} \bar{\mu}_L \sigma^{\mu\nu} e_R \, F_{\mu\nu} + \dots\,, \tag{6.75}$$

where the dots denote terms we don't care about right now. This operator causes the decay $\mu^- \to e^- + \gamma$ which means it violates Lepton flavor symmetry (an accidental symmetry again of the Standard Model). This process is actively pursued but so far has never been measured. The current best limit is from an experiment at PSI in Switzerland and obtains for the branching ratio of this process

$$\text{B.R.}(\mu^- \to e^- + \gamma) < 4.2 \cdot 10^{-13} \tag{6.76}$$

It's a good exercise to work this out and you should obtain that this implies $\Lambda > 10^7$ GeV or so. So if we want beyond-the-SM physics to live at scales of a few TeV, it better conserve lepton flavor symmetry!

CP violation. If we set $p = r = 1$ we get instead

$$\begin{aligned} \mathcal{L}_{\text{EDM}} &= \frac{c_W v}{\Lambda^2\sqrt{2}} \left[C^{(6)}_{eB}\right]_{11} \bar{e}_L \sigma^{\mu\nu} e_R \, F_{\mu\nu} + \text{h.c.} \\ &= \frac{c_W v}{\Lambda^2\sqrt{2}} \left\{ \text{Re} \left[C^{(6)}_{eB}\right]_{11} \bar{e} \sigma^{\mu\nu} e \, F_{\mu\nu} + \text{Im} \left[C^{(6)}_{eB}\right]_{11} \bar{e} \sigma^{\mu\nu} i\gamma^5 e \, F_{\mu\nu} \right\}. \end{aligned} \tag{6.77}$$

There are two terms, one proportional to the real part of the Wilson coefficient and one to the imaginary part. The real part is a correction to the magnetic dipole moment of the electron. We will not discuss this right now (but see below for the discussion of the muonic case). The second term turns out to be a correction to the electric dipole moment of the electron (you can see this by taking a non-relativistic limit of the expressions and you will find a term $d_e \, \vec{\sigma} \cdot \vec{E}$ where $\vec{\sigma}$ is the electron spin and $\vec{E}$ the electric field. We obtain

$$d_e = \frac{\sqrt{2} c_W v}{\Lambda^2} \text{Im} \left[C^{(6)}_{eB}\right]_{11} . \tag{6.78}$$

An electric dipole moment of a fundamental particle such as an electron signals the violation of CP symmetry. This can be most easily understood from performing a CP transformation on the Lagrangian or by noticing that $\vec{\sigma} \cdot \vec{E}$ flips under time-reversal symmetry (spin flips but electric field does not) and then using the fact that CPT is conserved in the SM. So T violation implies CP violation.

There are many experiments all over the world that aim to detect the electric dipole moment (EDM) of the electron. In fact, there is an experiment in the Netherlands (a collaboration between the UvA-VU-RUG).

Nobody has every measured such a property but there are extremely strong limits:

$$d_e^{\rm exp} < 4.1 \cdot 10^{-30}\, e\,{\rm cm} \simeq \frac{1}{0.5 \cdot 10^{16}\,{\rm GeV}}\,. \tag{6.79}$$

This field is improving rapidly and the limits were a thousand times weaker than a decade ago. This bound tells us

$$\frac{\Lambda}{\sqrt{{\rm Im}\left[C_{eB}^{(6)}\right]_{11}}} > \sqrt{\frac{\sqrt{2} c_W v}{d_e^{\rm exp}}} > 1.2 \cdot 10^9\,{\rm GeV}\,. \tag{6.80}$$

Again this limit is very impressive, and it essentially tells us that if we want new physics at a scale of a few TeV, then we have to be very careful with the amount of CP violation in such a theory. Otherwise we would create too large EDMs that should have already been measured !

An example of a theory that produces EDMs is supersymmetry. If we integrate out the supersymmetric particles you can obtain a one-loop contribution to the [Im $\left[C_{eB}^{(6)}\right]_{11}$] coefficient. Roughly, ignoring factors of 2 etc, you would obtain

$$d_e = \frac{v y_e \alpha_{\rm em}}{(4\pi) M_{\rm SUSY}^2} \sin\phi_{CP}\,, \tag{6.81}$$

where $M_{\rm SUSY}$ is the typical mass scale of the supersymmetric particles and ϕ_{CP} a phase appearing in the Lagrangian. y_e is the electron Yukawa coupling. If we compare this to Eq. (6.78) we can identify $\Lambda \sim M_{\rm SUSY}$ which makes sense, and

$$[{\rm Im}\left[C_{eB}^{(6)}\right]_{11}] \sim \frac{y_e \alpha_{\rm em}}{4\pi} \sin\phi_{CP}\,, \tag{6.82}$$

which indicates that [Im $\left[C_{eB}^{(6)}\right]_{11}$] is much smaller than 1. There is a Yukawa suppression $y_e \sim m_e/v \sim 10^{-6}$ and a one-loop factor $\alpha_{\rm em}/(4\pi) \sim 10^{-3}$. If we assume that CP-odd phase is $\mathcal{O}(1)$ (as it is in the SM) then the EDM limits give

$$M_{\rm SUSY} > \sqrt{\frac{v y_e \alpha_{\rm em}}{(4\pi) d_e^{\rm exp}}} \simeq 4 \cdot 10^4\,{\rm GeV} = 40\,{\rm TeV}\,. \tag{6.83}$$

Before the LHC started there were high hopes that the LHC would discover supersymmetry with $M_{\rm SUSY} \sim 1$ TeV. The EDM limits were already somewhat in tension with this and this was called the SUSY CP-problem as TeV-scale supersymmetry was only allowed if the CP-odd phases were tuned to small values and there was no good motivation for this. In any case, the matching to supersymmetry shows that Wilson coefficients do not have to be $\mathcal{O}(1)$ at all! In this case, we get large suppression from small dimensionless couplings (the electron Yukawa and the fine-structure constant).

The muon anomalous magnetic moment. If we set $p = r = 2$ we now get

$$\mathcal{L}_{g-2} \;=\; \frac{c_W v}{\Lambda^2 \sqrt{2}} \left\{ {\rm Re}\left[C_{eB}^{(6)}\right]_{22} \bar\mu \sigma^{\mu\nu} \mu\, F_{\mu\nu} + {\rm Im}\left[C_{eB}^{(6)}\right]_{22} \bar\mu \sigma^{\mu\nu} i\gamma^5 \mu\, F_{\mu\nu} \right\}\,. \tag{6.84}$$

We'll look at the first term which is a correction to the muon magnetic dipole moment. Unlike the EDM, the magnetic dipole moment (MDM) does not violate CP. We will call the magnetic dipole moment μ which should not be confused with the symbol for the muon. Already from QED we obtain a correction to the magnetic dipole moment of the muon is given by $\mu = eg/(2m_\mu)$ where m_μ is the muon mass and g is called the g-factor. At tree-level $g = 2$ but it gets correction at higher loops which are parametrized through

$$a_\mu = \frac{g-2}{2}\,, \tag{6.85}$$

where a_μ is called the anomalous magnetic moment of the muon. You have probably calculated in a QFT class the anomalous magnetic dipole moment of the muon at one loop in QED which gives $a = \alpha_{\rm em}/(2\pi)$ (note that at this order in the expansion, this result is also valid for the electron and the tau lepton). Nowadays theorists have computed a_μ up to 5 loops in QED + 2 loop electroweak corrections + QCD corrections. You might be surprised that QCD enters a muonic quantity but remember that at high-loop order you can draw diagrams with quark loops which then can exchange gluons.

About 2-3 years ago, the community made the following prediction for a_μ

$$a_\mu = (116\,591\,810 \pm 43) \cdot 10^{-11} \tag{6.86}$$

with a stunning accuracy (note the 116 digits arise simply from $\alpha_{\rm em}/(2\pi) \simeq 0.00116...$). At the same time the $g-2$ collaboration performed a new measurement of the muon anomalous magnetic moment by storing muons in a storage ring. The experiment confirmed the result of an earlier experiment and found

$$a_\mu^{\rm exp} = (116\,592\,061 \pm 41) \cdot 10^{-11}\,, \tag{6.87}$$

which looks close enough to the theoretical prediction. But if you subtract the two you obtain

$$\Delta a_\mu = (251 \pm 59) \cdot 10^{-11}\,, \tag{6.88}$$

and thus signals a 4 standard deviation discrepancy with the predictions of the Standard Model. People have been extremely excited about this result.

Before dosing the flames, let us fist try to figure out how to explain this result with our SM-EFT operator. Eq. (6.84) corrects the muon dipole moment by

$$\mu_\mu = \frac{\sqrt{2} c_W v}{\Lambda^2} \mathrm{Re}\left[C_{eB}^{(6)}\right]_{22}\,, \tag{6.89}$$

and thus a correction to a_μ given by

$$a_\mu^{\rm SMEFT} = \frac{2\sqrt{2} c_W v m_\mu}{\Lambda^2} \mathrm{Re}\left[C_{eB}^{(6)}\right]_{22}\,, \tag{6.90}$$

and if we want to explain the discrepancy in Eq. (6.88), we require

$$\frac{\Lambda}{\sqrt{\mathrm{Re}\left[C_{eB}^{(6)}\right]_{22}}} = \sqrt{\frac{2\sqrt{2} c_W v m_\mu}{\Delta a_\mu}} = 1.5 \cdot 10^5\,\mathrm{GeV}\,. \tag{6.91}$$

While this seems high enough and well outside the range of LHC physics we have to remember again that $\mathrm{Re}\left[C_{eB}^{(6)}\right]_{22}$ is not necessarily $\mathcal{O}(1)$. In fact, in many models of new physics as we saw for the electron EDM we get

$$\mathrm{Re}\left[C_{eB}^{(6)}\right]_{22} \sim \frac{y_\mu \alpha_{\rm em}}{4\pi} \simeq 2 \cdot 10^{-7}\,, \tag{6.92}$$

which would lower the scale to $\Lambda \simeq 100$ GeV which is way too low. Such particles should have been seen at the LHC and earlier experiments.

The community has therefore focused on scenarios with a so-called chiral enhancement. A famous example are leptoquarks (you can google that) in which you get a less severe scaling

$$\text{Re}\left[C_{eB}^{(6)}\right]_{22} \sim \frac{y_t}{(4\pi)^2}\,, \tag{6.93}$$

where $y_t \sim 1$ is the Top Yukawa coupling. In those cases you would obtain $\Lambda \simeq 10$ TeV. The LHC experiments are carefully looking for these hypthetical leptoquarks and right now they are excluded up to masses with a few TeV. So the verdict is still out.

Finally, we have to say that the muon $g-2$ anomaly might be due to a wrong assessment of the theory prediction in Eq. (6.86). Since the community presented this number, there have been groups, who computed the QCD corrections in a different way using a technique called Lattice QCD. Their findings seem to indicate that the theory prediction is a bit larger and thus moving towards the experimental value. It might thus be that there is no anomaly at all. Again the verdict is still out on this. Exciting times.

6.10 And the list goes on

We have only scratched the surface of the SM-EFT dimension-six operators. People have investigated many more operators and related observables. For instance, an operator such as $H^\dagger H G^a_{\mu\nu} G^{a,\,\mu\nu}$ modifies the production of Higgs bosons at the LHC. Careful measurements then allow us to constrain the associated Wilson coefficients. Operators with top quarks modify top-quark production and decay processes at high-energy collisions and a large amount of data exists for this. Juan is developing tools and methods to analyze all this data, and related processes, in terms of the SM-EFT Lagrangian. I have been interested recently in how the SM-EFT can modify the extraction of the CKM parameters of the Standard Model. The SM predicts that the CKM matrix in Eq. (6.51) is unitary. This for instance implies

$$|V_{ud}|^2 + |V_{us}|^2 + |V_{ub}|^2 = 1\,, \tag{6.94}$$

but the most accurate extractions of these elements find a number smaller than 1 by about 3 to 4 standard deviations. Me and my group and collaborators are trying to investigate how this could potentially be explained by beyond-the-SM physics, and we are using the SM-EFT framework to do so.

7 Chiral Perturbation Theory

So far in this course we have focused mainly on EFTs where heavy degrees of freedom can be integrated out by hand by performing a perturbative matching calculation. A good example of this is the LEFT, where the heavy SM particles (W, Z, top, Higgs) are decoupled. EFTs can also be very powerful tool, although it is slightly more complicated, in non-perturbative theories. A great example is low-energy QCD where perturbation theory in the strong coupling constants g_s (or perhaps better in $\alpha_s = g_s^2/(4\pi)$) breaks down. At first sight this seems a very big obstacle. How can we compute anything involving the relevant low-energy degrees of freedoms (hadrons such as mesons and baryons) if we cannot use perturbation theory?

During the 50's and 60's of the twentieth century physicists worked out complicated methods to still make predictions for hadronic interactions. These techniques went under the name of 'current algebra' and, I can say this because I worked through some of this stuff, it is frankly messy, complicated, and unclear. Weinberg (who else) figured out a way to understand the current algebra results in a much neater way by developing a theory called chiral perturbation theory. Nowadays, we can understand why this works because chiral perturbation theory is the low-energy EFT of QCD. In these notes we are going to discuss how to develop and use chiral perturbation theory (χPT). This topic is somewhat more complicated than earlier EFTs and we will not have time to discuss all details. If you are interested very good resources are available. There is a dedicated chapter in Volume 2 of Weinberg's QFT book which is great but unfortunately uses a different notation then most of the field. Another excellent, but somewhat long-winded, is the χPT book by Scherer and Schindler. Most stuff I write here follows that book but I avoid many technical details.

7.1 Global symmetries of QCD

Let us consider the QCD Lagrangian at relatively low energies. We assume we integrate out the heavy SM degrees of freedom, and, at first, neglect the electromagnetic and weak interactions. At an energy scale of 1 GeV or so, we obtain the Lagrangian

$$\mathrm{L}_{\mathrm{QCD}} = -\frac{1}{4}G^a_{\mu\nu}G^{a\,\mu\nu} + \bar{q}\left(i\not{D} - M_q\right)q + \mathcal{O}\left(\frac{1}{m_Q}\right)\,, \tag{7.1}$$

where we kept only the light quark fields

$$q = \begin{pmatrix} u \\ d \\ s \end{pmatrix}\,, \qquad D_\mu q = \left(\partial_\mu - i\frac{g_s}{2}\,A^a_\mu \lambda^a\right)q\,. \tag{7.2}$$

So we have combined the three light quarks into a single object q, and each light quark feels the same strong interaction. As mentioned we have omitted electromagnetic/weak interactions for now. M_q is a diagonal and real 3×3 matrix: $M_q = \mathrm{diag}\,(m_u\,, m_d,\, m_s)$. The $\mathcal{O}\,(1/m_Q)$ terms denote higher-dimensional terms that are suppressed by the heavy quark masses and we neglect them here.

We keep only the light quarks u, d, and s because they have masses well below 1 GeV. In fact, for the purpose of these notes we are going to work in the two-flavored approximation where we only keep the lightest 2 quarks $m_{u,d} \ll m_s$. This will simplify our life somewhat but we do not lose any essential insights. So from now one we will use

$$q = \begin{pmatrix} u \\ d \end{pmatrix}\,, \qquad D_\mu q = \left(\partial_\mu - i\frac{g_s}{2}\,A^a_\mu \lambda^a\right)q\,, \tag{7.3}$$

and $M_q = \text{diag}(m_u\,, m_d)$. Let us move to a basis of chiral quark fields: $q = q_L + q_R$ and we obtain

$$\mathrm{L}_{\mathrm{QCD}} = -\frac{1}{4} G^a_{\mu\nu} G^{a\,\mu\nu} + \bar{q}_L i \not{D} q_L + \bar{q}_R i \not{D} q_R - \bar{q}_R M_q q_L - \bar{q}_L M_q^\dagger q_R \,. \tag{7.4}$$

(because we said M_q is real and diagonal, in principle $M_q^\dagger = M_q$ but writing it like this makes our life a bit easier later).

Now here comes the crux: if we neglect M_q the Lagrangian becomes very simple (just the kinetic terms of gluons and left- and right-handed quarks). You might protest and argue that there is no clear reason to neglect the quark masses. It will turn out that light quark masses are small compared to the mass scale that is generated by QCD dynamically. We will see that up and down quark masses are of the order of a few MeV, but protons and neutrons have masses of 1 GeV roughly. In this light, it is not crazy to neglect quark masses. Anyway for now let's just neglect them and see what this will teach us.

After neglecting the quark masses, the Lagrangian has 4 accidental global symmetries. We have two independent $U(1)$ symmetries $q_L \to e^{i\theta_L} q_L$ and $q_R \to e^{i\theta_R} q_R$, and two independent $SU(2)$ symmetries

$$q_L \to U_L q_L = e^{\frac{i}{2}\tau^a \theta^a_L} q_L \,, \tag{7.5}$$
$$q_R \to U_R q_R = e^{\frac{i}{2}\tau^a \theta^a_R} q_R \,, \tag{7.6}$$

where $a = \{1, 2, 3\}$ (if we had included strange quarks these symmetries would be $SU(3)$ instead of $SU(2)$). Now for each global symmetry we can compute a Noether current and an associated conserved charge. The Noether currents are easily derived (remember $J^\mu = \sum_n (\delta\mathcal{L}/(\delta\partial_\mu \Psi_n))\delta\Psi_n$ where the sum runs over all fields) and we obtain for the $U(1)$ symmetries

$$L^\mu = \bar{q}_L \gamma^\mu q_L \,, \qquad R^\mu = \bar{q}_R \gamma^\mu q_R \,, \tag{7.7}$$

which are conserved by the classical equations of motion $\partial_\mu L^\mu = \partial_\mu R^\mu = 0$. For the $SU(2)$ symmetries we obtain the Noether currents

$$L^{\mu,a} = \frac{1}{2} \bar{q}_L \gamma^\mu \tau^a q_L \,, \qquad R^{\mu,a} = \frac{1}{2} \bar{q}_R \gamma^\mu \tau^a q_R \,, \tag{7.8}$$

which are also conserved. It is useful to combine the currents into vector and axial-vector currents

$$V^\mu = L^\mu + R^\mu \,, \qquad A^\mu = L^\mu - R^\mu \,, \qquad V^{\mu,a} = L^{\mu,a} + R^{\mu,a} \,, \qquad A^{\mu,a} = L^{\mu,a} - R^{\mu,a} \,. \tag{7.9}$$

To each current there belongs a conserved charge. For example, for the singlet vector current we obtain

$$Q_V = \int d^3x\, V^0 = \int d^3x\, \bar{q}(x)\gamma^0 q(x) = \int d^3x\, q(x)^\dagger q(x) \,, \tag{7.10}$$

which can be interpreted as the number operator of quark fields (it counts the number of quarks essentially). Similarly we can compute charges for the other symmetry groups, e.g. for $SU(2)_{V,A}$

$$\begin{aligned} Q^a_V &= \int d^3x\, V^{0,a} == \frac{1}{2} \int d^3x\, q(x)^\dagger \tau^a q(x) \,, \\ Q^a_A &= \int d^3x\, A^{0,a} == \frac{1}{2} \int d^3x\, q(x)^\dagger \gamma^5 \tau^a q(x) \,. \end{aligned} \tag{7.11}$$

These charges commute with the Hamiltonian of massless QCD, e.g. $[\mathcal{H}_{\mathrm{QCD},m_q=0}, Q_V] = 0$, and thus are time-independent: they are conserved charges.

Now, let's study if we see these symmetries and charges back in the spectrum of QCD. We classify hadrons into objects with baryon number zero ($B = 0$), mesons, and $B = 1$, nucleons, and larger objects $B > 1$, atomic nuclei. Baryon number is conserved in nature (see the SM-EFT lectures) and this is related to the conservation of Q_V. But what about the axial $U(1)$ symmetry related to A_μ? This we do not see back in the spectrum, it would imply that there is a second conserved charge related to hadrons with opposite parity. This problem was solved by Jackiw, Adler, Bell, and 't Hooft who noticed that the $U(1)_A$ symmetry is only a classical symmetry but is broken by quantum effects. A classical symmetry that is broken at the quantum level is called *anomalous*. We will not discuss anomalous symmetries in this lecture, instead we will simply say that the anomaly ensure that the Noether current $\partial_\mu A^\mu \neq 0$. So we actually should not see this symmetry in the hadron spectrum! This is a fascinating story with a lot of interesting physics so you are encouraged to read up on this.

Now let's move to the $SU(2)$ symmetries. You can compute that $[Q_V^a, Q_V^b] = i\epsilon^{abc}Q_V^c$, and since Q_V^a commutes with the Hamiltonian, we can copy everything we know from angular momentum $\hat{\boldsymbol{L}}$ in quantum mechanics in systems with rotational invariance. That is, in QM in a central potential (where $[\hat{\boldsymbol{L}}, \hat{H}] = 0$) we can label states by angular momentum l, where $l = 0, \frac{1}{2}, 1, \dots$, and states with a given l have a quantum number m that ranges from $-l$ to $+l$ in unit steps. Rotational symmetry implies that states within a multiplet (same l but different m) are degenerate. We have exactly the same thing here but now for Q_V^a instead of $\hat{\boldsymbol{L}}$. This suggest we can label states by a quantum number I we call isospin and $l = 0, \frac{1}{2}, 1, \dots$ and for each I there is an m_I that ranges from $-I$ to I.

But this is exactly what we observe in nature! For instance, the lowest-mass baryons (the proton and neutron) have almost identical mass $(m_n - m_p)/(m_n + m_p) \simeq 10^{-3}$ and, as far as strong interaction physics is conserved, have identical properties (they of course have a different charge but we are working in the limit where we can neglect electromagnetism). We can thus combine protons and neutrons into a nucleon field with $I = 1/2$ (read: isospin 1/2) whose $m_I = +1/2$ component is the proton and the $m_I = -1/2$ is the neutron. Similarly, the lowest mass mesons are the pions (π^0, π^+, π^-) with almost identical mass $m_\pi \simeq 135$ MeV. The three pions can be seen as the components of an $I = 1$ object. Isospin is a pretty good symmetry of nature and this is reflected, for example, by the fact that pion-neutron and pion-proton scattering gives very similar results (up to electromagnetic corrections).

So far so good. Two global symmetries of massless QCD are also seen in the hadron spectrum. But what about $SU(2)_A$? Of this we see no immediate trace in the low-energy spectrum. It would imply that next to the nucleon doublet there should be another doublet, with almost the same mass, but opposite parity. This is not observed. Of course, perhaps, like $U(1)_A$, the $SU(2)_A$ symmetry is anomalous but this turns out to be false (for reasons I cannot explain here). So what is going on? If a symmetry appears in the Lagrangian but is not manifest, it might be that the symmetry is spontaneously broken by the ground state[7]. If you are not familiar with this, please study the next subsection, otherwise just skip it.

7.1.1 Spontaneously broken symmetry recap

Just to remind you, let us study a simple example of spontanously broken global symmetries. Let's consider a theory with a single complex scalar ϕ

$$\mathcal{L} = (\partial_\mu \phi)(\partial_\mu \phi)^\dagger + \mu^2|\phi|^2 - \lambda|\phi|^4 \tag{7.12}$$

[7]We saw this already in the SM lecture where the minimum of the Higgs potential, v, broke the $SU(2)_L$ gauge symmetry. In this case we are talking not about a gauge (local) symmetry but a global symmetry).

which is invariant under global $U(1)$ transformations $\phi \to e^{i\alpha}\phi$. Here μ^2 and λ are just two real parameters. The potential of this theory is given by

$$V(\phi) = -\mu^2|\phi|^2 + \lambda|\phi|^4 \,. \tag{7.13}$$

We must have $\lambda > 0$ otherwise the theory is not bounded from below at large values of ϕ. If $\mu^2 < 0$ then this potential has a minimum at $\langle|\phi|\rangle = 0$. We can then expand our field around the minimum $\phi = (0 + \phi_1 + i\phi_2)/\sqrt{2}$ where $\phi_{1,2}$ are real, and our Lagrangian becomes

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi_1)(\partial^\mu\phi_1) + \frac{1}{2}(\partial_\mu\phi_2)(\partial^\mu\phi_2) - \frac{|\mu^2|}{2}\left(\phi_1^2 + \phi_2^2\right) - \frac{\lambda}{4}\left(\phi_1^2 + \phi_2^2\right)^2 \,, \tag{7.14}$$

describing a theory of 2 scalars with equal mass $m_1 = m_2 = |\mu|$ and interactions proportional to λ. In this case, the symmetry of the Lagrangian is explicitly realized in the ground states because $\langle|\phi|\rangle = 0$ is invariant under the $U(1)$ symmetry.

The picture is very different if we consider the case $\mu^2 > 0$. In this case, the minimum occurs at non-zero value

$$\langle|\phi|^2\rangle = \frac{\mu^2}{2\lambda} \equiv \frac{v^2}{2} \,. \tag{7.15}$$

There are many degenerate ground states then $\langle\phi\rangle = ve^{i\beta}/\sqrt{2}$ for any value of β. If we pick one minimum, say $\beta = 0$ then the minimum $\langle\phi\rangle = v/\sqrt{2}$ is no longer invariant under $U(1)$ transformations. If we expand around the minimum we can write $\phi = (v + \phi_1 + i\phi_2)/\sqrt{2}$ but it is more transparant to instead write

$$\phi = \frac{1}{\sqrt{2}}(v + \rho(x))e^{i\theta(x)/v} \,, \tag{7.16}$$

where we have traded the real fields $\phi_{1,2}$ for the real fields ρ and θ. If we put this into Eq. (7.12) we obtain

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\rho)(\partial^\mu\rho) + \frac{1}{2}(\partial_\mu\theta)(\partial^\mu\theta)\left(1 + \frac{\rho}{v}\right)^2 - \mu^2\rho^2 - \lambda v\rho^3 - \frac{\lambda}{4}\rho^4 \,. \tag{7.17}$$

This Lagrangian describes a massive scalar ρ with $m_\rho = \sqrt{2}\mu$ and a massless scalar θ. The massless scalar interacts with the ρ field but only through derivative couplings (through the $\rho(\partial_\mu\theta)^2$ and $\rho^2(\partial_\mu\theta)^2$ terms). Such a massless scalar with derivative couplings is a generic feature of spontaneously broken global symmetries and is called a Goldstone boson.

Goldstone's theorem says that if a system has a symmetry group G that is broken to a subgroup H in the groundstate, then there appears a Goldstone boson for each broken generator. That is, the number of Goldstone bosons is given by $\dim[G] - \dim[H]$ (this is the dimension of the coset space G/H which contain the elements of G that do not appear in H.). In the above example we have $\dim[G] = 1$ and the ground-state has no symmetry left and thus $\dim[H] = 0$. We thus expect 1 Goldstone boson as we found explicitly.

It is also useful to see how the Goldstone boson is parametrized in Eq. (7.16) through the exponential $e^{i\theta(x)/v}$. This is exactly the form of spontaneously broken $U(1)$ transformations! This is not an accident: the Goldstone bosons 'live' in the space of the broken generators. In this example, that is the $U(1)$ group.

7.1.2 Spontaneously broken chiral symmetry

We observed that massless QCD has a global $SU(2)_V \otimes SU(2)_A$ symmetry, but the hadron spectrum only has an $SU(2)_V$ symmetry (called isospin symmetry). Let's assume that this occurs because the ground-state of QCD (which is a tricky non-perturbative thing) spontanously breaks $SU(2)_A$ symmetry. Goldstone's theorem then tells us we should expect three Goldstone bosons associated to the broken generators of $SU(2)_A$.

We can also get some intuition from this with a simple argument. We know that the Hamiltonian of massless QCD (we called this $H_{\text{QCD},m_q=0}$ but let me call it $\mathcal{H}$ to save some writing) commutes with the all Noether charges of the full group $SU(2)_V \otimes SU(2)_A$. That is

$$[\mathcal{H},\, Q^a_V] = [\mathcal{H},\, Q^a_A] = 0\,. \tag{7.18}$$

Let's denote the groundstate of QCD as $\mathcal{H}|0\rangle = 0$. We assume now that the vacuum is left in tact by the subgroup H (in our case $H = SU(2)_V$) but not by the coset space G/H (in our case $SU(2)_A$). This implies

$$Q^a_A|0\rangle \neq 0\,, \tag{7.19}$$

but these states, while different from the vacuum state, are degenerate with the vacuum because

$$\mathcal{H}Q^a_A|0\rangle = Q^a_A\mathcal{H}|0\rangle = 0\,, \tag{7.20}$$

which implies that for each Q^a_A there has to be a zero-energy state[8] with the quantum numbers of Q^a_A (in our case these are pseudoscalar fields because of the γ^5 appearing in Q^a_A). A zero-energy states implies a massless particle.

So we are expecting three massless Goldstone bosons to appear in low-energy QCD if indeed $SU(2)_L \times SU(2)_R \to SU(2)_V$ is spontaneously broken. These bosons should have negative parity. The low-energy spectrum does not have massless particles.... So is all hope lost? Well, perhaps we should not really expect massless particles because QCD does not really have $SU(2)_L \times SU(2)_R$ global symmetry! This is only true if quarks had been massless. So perhaps we have been too strict. If we look again at the hadron spectrum we do identify three particles that are much lighter than any other hadron: the pions. There are three of them $\pi^\pm$, π^0 and they have almost degenerate masses $m_\pi \simeq 135$ MeV which is much smaller than say the proton or nucleon mass. Also the pions have negative intrinsic parity! So let us take this as our working hypothesis: pions are the Goldstone bosons of the spontaneously broken $SU(2)_A$ and they would have been massless had we lived in a world where quarks were massless. We'll later make this more exact.

We would like to write down an EFT that describes the physics of the pions. From an EFT point of view, all other hadrons are relatively heavy and can be integrated out and we just keep the pions as our degrees of freedom. The main challenge right now is to find a suitable parametrization of the pion fields and figure out how the pions transform under $SU(2)_L$ and $SU(2)_R$. Once we know this, we are in business!

The derivation of the transformation of Goldstone bosons can be obtained from a construction develop by Callan, Coleman, Wess, and Zumino (CCWZ). This construction is a bit formal and involves some group theory arguments. It is not very complicated but goes a bit beyond what we can cover here. Instead I will try to give an intuitive argument. Remember that the Goldstone bosons correspond to transformations in the spontaneously broken direction. Basically we need to find a parametrization for an element of $SU(2)_L \otimes SU(2)_R/SU(2)_V$ where $SU(2)_V$ is the unbroken subgroup. We will use the notation where if we perform an $SU(2)_L$ and $SU(2)_R$ transformation we write this as (L, R) where L and R are $SU(2)$ transformations. We then parametrize an element in $SU(2)_L \otimes SU(2)_R/SU(2)_V$ through

$$(L, R)(V^{-1}, V^{-1})\,, \tag{7.21}$$

[8]This argument is actually too naive, the states that are obtained from $Q^a_A|0\rangle$ have infinite norm and thus not proper states. A more solid argument can be found in Scherer/Schindler or Weinberg volume II. I will not go into this here.

where if we choose $V = L$ we obtain

$$(L, R)(L^{-1}, L^{-1}) = (1, RL^{-1}) = (1, RL^\dagger) \equiv (1, U)\,. \tag{7.22}$$

We will then parametrize our pions through U. But U is just the product of 2 $SU(2)$ matrices and is thus an $SU(2)$ element itself. We thus can parametrize

$$U = e^{i\Pi/f_\pi} = e^{i\tau^a\pi^a/f_\pi}\,, \tag{7.23}$$

where f_π is a constant with mass dimension 1 that plays a similar role as v in Eq. (7.16). f_π is called the pion decay constant although right now it is not clear why it has that name. It has a numerical value $f_\pi \simeq 92$ MeV. Note that we will use the notation $\Pi = \pi^a\tau^a$

So let's do another transformation on Eq. (7.22), it will transform as

$$(L, R)(1, U)(L^{-1}, L^{-1}) = (1, RUL^\dagger)\,, \to U \to RUL^\dagger\,. \tag{7.24}$$

We can check that this makes sense. For instance $U = 1 + i\Pi/f_\pi + \mathcal{O}(\pi^2)$ and thus the ground state is the state with no pion fields $U_0 = 1$. This ground state transforms as $U_0 \to RU_0L^\dagger = RL^\dagger$. This is invariant if $R = L$, corresponding to the isospin subgroup $SU(2)_V$, but not for $SU(2)_A$ transformations which is exactly what we want. We can also figure out how the pions transform under $SU(2)_V$ if we expand

$$VUV^\dagger = V(1 + i\Pi/f_\pi + \mathcal{O}(\pi^2))V^\dagger = 1 + \frac{i}{f_\pi}V\Pi V^\dagger + \dots\,, \tag{7.25}$$

and thus $\pi \to V\pi V^\dagger$ and transforms in the adjoint under isospin rotations. This explains why pions form an isospin $I = 1$ state. More explicitly under infinitesimal transformations $V = 1 + i(\theta_L^a + \theta_R^a)\tau^a + \dots$ we obtain

$$\pi^a \to \pi^a + i(\theta_L^b + \theta_R^b)[\tau^b,\, \tau^c]\pi^c = \pi^a - 2\epsilon^{abc}(\theta_L^b + \theta_R^b)\pi^c + \dots\,. \tag{7.26}$$

But under the $SU(2)_A$ transformations we get complicated properties (for instance something like $\pi^a \to \pi^a - 2if_\pi(\theta_L^a - \theta_R^a) + \dots$.

7.2 The leading-order χPT Lagrangian

We can now construct the EFT for pions. The fields are parametrized through U which transforms as $U \to RUL^\dagger$. And note that U is dimensionless while the pion fields $[\pi^a] = 1$ have dimension.

We can build invariant terms such as $\mathrm{Tr}[UU^\dagger]$, where Tr denotes taking the trace of the matrix. This transforms as

$$\mathrm{Tr}[UU^\dagger] \to \mathrm{Tr}[RUL^\dagger LLU^\dagger R^\dagger] = \mathrm{Tr}[UU^\dagger]\,, \tag{7.27}$$

where in the last step we used the cylic property of traces $\mathrm{Tr}[AB\dots YZ] = \mathrm{Tr}[ZAB\dots Y]$. Unfortunately this term is useless because $UU^\dagger = 1$ and thus there are no interesting terms in here.

To get the first interesting term we need to use two derivatives. For instance we can construct

$$\mathcal{L}_2 = C_2\mathrm{Tr}[(\partial_\mu U)(\partial^\mu U^\dagger)]\,, \tag{7.28}$$

which is invariant. Here C_2 is the coefficient in front of the interaction and is essentially a Wilson coefficient in χPT. However, for historic reasons in χPT these coefficients are called low-energy constants (LECs). In this case we have $[C_2] = 2$. Now let us expand our Lagrangian in pion fields. With a bit of algebra you will obtain

$$\mathcal{L}_2 = C_2 \left[\frac{2(\partial_\mu \pi)^2}{f_\pi^2} - \frac{2}{3 f_\pi^4} \left(\pi^2 (\partial_\mu \pi)^2 - (\pi \cdot \partial_\mu \pi)^2 \right) \right] + \mathcal{O}(\pi^6)\,, \tag{7.29}$$

where $(\partial_\mu \pi)^2 = (\partial_\mu \pi^a)(\partial^\mu \pi^a)$. We want to normalize the first term in order to get the usual kinetic term for scalar fields so we are forced to set $C_2 = f_\pi^2/4$ and we obtain

$$\mathcal{L}_2 = \frac{1}{2}(\partial_\mu \pi)^2 - \frac{1}{6 f_\pi^2} \left(\pi^2 (\partial_\mu \pi)^2 - (\pi \cdot \partial_\mu \pi)^2 \right) + \mathcal{O}(\pi^6)\,. \tag{7.30}$$

So we see that this not only describes three massless scalar fields $\pi^{1,2,3}$ but also their interactions! And the coupling strength of their interactions is also fixed in terms of the pion decay constant (in the exercises you will show how to obtain a value for f_π by studying weak decays of pions).

It is interesting to study the scattering of pions. If we add electromagnetism to the game, we will identify that $\pi^\pm = (\pi_1 \pm i\pi_2)/\sqrt{2}$ but let's for now just consider the three pions as separate. One thing we obtain from L_2 is that $\pi^0\pi^0 \to \pi^0\pi^0$ does not occur at this order in the theory (we will see that this does happen once we consider quark mass effects). Let's consider $\pi^0(p_a)\pi^0(p_b) \to \pi^1(p_c)\pi^1(p_d)$ as an example. We will get

$$\begin{aligned} i\mathcal{A}_{\rm LO}\left(\pi^0(p_a)\pi^0(p_b) \to \pi^1(p_c)\pi^1(p_d)\right) &= \frac{1}{6 f_\pi^2}\left[4p_a\cdot p_b + 4p_c\cdot p_d + 2p_a\cdot p_c + 2p_a\cdot p_d + 2p_b\cdot p_c + 2p_b\cdot p_d\right] \\ &= \frac{1}{6 f_\pi^2}\left[4s - 2t - 2u\right] = \frac{s}{f_\pi^2}\,, \end{aligned} \tag{7.31}$$

where we used Mandelstam variables $s = (p_a+p_b)^2 = 2p_a\cdot p_b$ (for massless particles), $t = (p_a-p_c)^2 = -2p_a\cdot p_c$, and $u = (p_a - p_d)^2 = -2p_a\cdot p_d$, and $s+t+u = 0$ (from four-momentum conservation). So we see that the amplitude of scattering is proportional to s which, in the center of the mass, is related to the total energy of the scattering process. This implies that for small energies, the scattering is weak! Of course we could have seen this already from the Lagrangian: the pion interactions involve derivatives and thus, at small energies, the pions only interact weakly. This will be crucial to set-up a consistent power counting for χEFT.

7.3 The role of quark masses

So far we have considered an idealized version of QCD where $SU(2)_L \times SU(2)_R$ is an exact symmetry. In reality this symmetry is broken by quark masses[9]. Let us look back at our original QCD Lagrangian

$$\mathrm{L}_{\rm QCD} = -\frac{1}{4} G^a_{\mu\nu} G^{a\,\mu\nu} + \bar{q}_L i\not{D} q_L + \bar{q}_R i\not{D} q_R - \bar{q}_R M_q q_L - \bar{q}_L M_q^\dagger q_R\,. \tag{7.32}$$

where $M_q = \mathrm{diag}(m_u,\, m_d)$. It is clear now that under the transformations $q_L \to L q_L$ and $q_R \to R q_R$ that the Lagrangian is not invariant because of the mass terms. But this is not a huge problem because quark masses are small. The way to account for them in the χPT Lagrangian is by using a little trick. We pretend that M_q is a field instead of a constant matrix. We call such a field a **spurion**. We then notice that QCD would have exact chiral symmetry if our spurion field would transform as

$$M_q \to R M_q L^\dagger\,. \tag{7.33}$$

The idea is now that we will build operators in the χPT Lagrangian that include the spurion M_q explicitly

[9] In fact, the symmetry is also broken by quark electromagnetic charges and by the dimension-six LEFT operators that arise from the weak interactions. It is relatively straightforward to extend χPT to also include these terms, but I will not do this in these notes. One of the exercises however will sketch how this works.

and that are $SU(2)_L \times SU(2)_R$ taking the transformation in Eq. (7.33) into account. Then once we have constructed the Lagrangian we just reduce M_q back to its constant form. In this way, we have included terms in χPT that explicitly break chiral symmetry but it breaks it in the same way as the symmetry-breaking terms in QCD. Ok this might be difficult to understand right now but let's just build the lowest-order terms that are linear in the symmetry-breaking quark mass term. We can construct one term without derivatives

$$\mathcal{L}_{M_q} = \frac{f_\pi^2 B}{2}\mathrm{Tr}[M_q^\dagger U + U^\dagger M_q]\,, \tag{7.34}$$

which is invariant. The LEC is conventionally written as $F_\pi^2 B/2$ where $[B] = 1$. To see what this term does, we now set $M_q = \mathrm{diag}(m_u,\, m_d)$ and expand the U matrix. I find it useful to write

$$M_q = \bar{m}\,\mathcal{I}_2 - \bar{m}\,\varepsilon\,\tau_3\,, \qquad \bar{m} = \frac{m_u + m_d}{2}\,, \qquad \varepsilon = \frac{m_d - m_u}{m_u + m_d}\,, \tag{7.35}$$

where $\mathcal{I}_2$ is the 2×2 identity matrix. Here $\bar{m}$ is the average quark mass which breaks $SU(2)_A$ but conserves $SU(2)_V$ (it conserves isospin because it treats up and down quarks in the same way), while the quark mass differences $\sim \varepsilon$ breaks both.

If you expand the Lagrangian we obtain

$$\begin{aligned}\mathcal{L}_{M_q} &= \frac{f_\pi^2 B}{2}\left[-\frac{2\bar{m}}{f_\pi^2}\pi^2 + \frac{\bar{m}}{6f_\pi^4}\pi^4 + \mathcal{O}(\pi^6)\right] \\ &= -\frac{m_\pi^2}{2}\pi^2 - \frac{m_\pi^2}{24 f_\pi^2}\pi^4 + \mathcal{O}(\pi^6)\,,\end{aligned} \tag{7.36}$$

where we have obtained the very important relation

$$m_\pi^2 = 2\bar{m}B = (m_u + m_d)B\,. \tag{7.37}$$

So we now understand why pions are not massless in nature despite their Goldstone nature! They pick up a mass which is proportional to the square root of the quark masses! We see that in the limit $\bar{m} \to 0$ the pions become massless as well. We also observe that at this order in the chiral Lagrangian, the quark mass difference plays no role! This is reflected by the observation that all 3 pions have the same mass. If you build terms with two powers of M_q this changes and you will then find isospin-breaking corrections that split the masses of the pion fields.

Let us make a brief tour into three-flavored χPT where we also include strange quarks. In this case, there are 8 Goldstone bosons which, in addition to the pions, are given by $K^\pm$, K^0, $\bar{K}^0, \eta$. If you do the exact same exercise for the quark mass terms but now $M_q = \mathrm{diag}(m_u,\, m_d,\, m_s)$ you would obtain the masses

$$m_\pi^2 = (m_u + m_d)B\,, \tag{7.38}$$

$$m_{K^0}^2 = (m_d + m_s)B \tag{7.39}$$

$$m_{K^\pm}^2 = (m_u + m_s)B \tag{7.40}$$

$$m_\eta^2 = \frac{1}{3}(4m_s + m_u + m_d)B\,. \tag{7.41}$$

These relations are known as the GellMann, Oakes, Renner relations and were known long before χPT was invented. They allow us to get a handle on quark mass ratios. For instance

$$\frac{m_u}{m_d} = \frac{m_{K^\pm}^2 - m_{K^0}^2 + m_\pi^2}{m_{K^0}^2 - m_{K^\pm}^2 + m_\pi^2} \simeq 0.66\,, \tag{7.42}$$

$$\frac{m_s}{m_d} = \frac{m_{K^\pm}^2 + m_{K^0}^2 - m_\pi^2}{m_{K^0}^2 - m_{K^\pm}^2 + m_\pi^2} \simeq 22\,, \tag{7.43}$$

so the up and down quark masses are similar, while the strange quark is significantly heavier. Nowadays, more elaborate fits to meson masses and scattering (and lattice-QCD calculations) we have $m_u \simeq 3$ MeV, $m_d \simeq 6$ MeV, and $m_s \simeq 130$ MeV. This then gives $B \simeq 2$ GeV.

We can also re-investigate pion-pion scattering but now include the pion mass. This has 2 changes: first of all Eq. (7.44) changes because of the kinematic relations $s = (p_a + p_b)^2 = 2m_\pi^2 + 2p_a \cdot p_b$ etc. And we now have the additional π^4 term in Eq. (7.36). If you work it out you will obtain

$$i\mathcal{A}_{\mathrm{LO}+m_q}\left(\pi^0(p_a)\pi^0(p_b) \to \pi^1(p_c)\pi^1(p_d)\right) \quad = \quad \frac{s - m_\pi^2}{f_\pi^2}\,, \tag{7.44}$$

so we see that quark mass effects are comparable to derivative interactions for $s \sim m_\pi^2$. That is, a two extra derivatives in an operator counts similarly as an insertion of $M_q \sim m_\pi^2$.

7.4 Higher-order terms

Since χPT is an EFT, we have to go beyond $\mathcal{L}_2$ by constructing terms with more derivatives. For instance, with 4 derivatives we can construct a term

$$\mathcal{L}_4 = L_1\left(\mathrm{Tr}[(\partial_\mu U)(\partial^\mu U^\dagger)]\right)^2 + L_2 \mathrm{Tr}[(\partial_\mu U)(\partial_\nu U^\dagger)]\,\mathrm{Tr}[(\partial^\mu U)(\partial^\nu U^\dagger)] + \ldots\,, \tag{7.45}$$

where the dots denote several other terms. These terms start with 4 pion fields and give corrections to pion-pion scattering that scale as $L_i p^4$ where p is the pion momentum. In an EFT, when you start adding higher-order terms we expect them to be suppressed by additional powers of the EFT breakdown scale. For χPT this scale is typically called Λ_χ, and we would expect, compared to C_2 in Eq. **??** that $L_2/C_2 \sim 1/\Lambda_\chi^2$ because there appear two additional derivatives in the Lagrangian. And thus $L_i \sim f_\pi^2/\Lambda_\chi^2$. So if we now add the extra, next-to-leading order (NLO) terms, from Eq. (7.45), to the scattering example from the previous section we would get, schematically,

$$i\mathcal{A}_{\mathrm{LO+NLO}} \simeq \frac{p^2}{f_\pi^2} + \frac{m_\pi^2}{f_\pi^2} + \frac{L_i p^4}{f_\pi^4} \simeq \frac{p^2}{f_\pi^2} + \frac{m_\pi^2}{f_\pi^2} + \frac{p^4}{f_\pi^2 \Lambda_\chi^2}\,, \tag{7.46}$$

where we have written $s \sim p^2$. The NLO correction is thus small compared to the LO terms as long as $p^2 \ll \Lambda_\chi^2$ and $p^4 \ll m_\pi^2 \Lambda_\chi^2$. To account for this we can assume $p \sim m_\pi$ (so pion masses are treated as the same scale as external momenta) and $p \ll \Lambda_\chi$. Higher-order terms are small as long as we consider low-energy processes. We'll study how to estimate Λ_χ in the next section.

To actually compute the impact of the terms in Eq. (7.45) we need to know the values of the LECs $L_{1,2}$. χPT does not tell us these values as they correspond to non-perturbative low-energy QCD dynamics. In practice what one does is to measure, say, pion-pion scattering at some energy. This then allows you to fit one of the LECs. You can then make predictions for all other processes (and energies, as long as we stay in the perturbative regime) where the same LEC enters. This turns out to be very powerful. A more modern approach is to compute the LECs with lattice-QCD, a non-perturbative numerical method to solve non-perturbative QCD. This requires large-scale supercomputer resources but allows on to compute the LECs directly from QCD.

7.5 The loop expansion

How can we determine what Λ_χ is? To do that it is useful to consider loop processes. Consider a loop correction to our scattering process $\pi^0(p_a)\pi^0(p_b) \to \pi^1(p_c)\pi^1(p_d)$, from the diagram in Fig. [x] (have to add this). We will just use the leading order vertices in Eq. (7.30). Each vertex contains 2 derivatives and let's look at the loop contributions where the derivatives act on the external pions. This will schematically give a correction to the amplitude

$$i\mathcal{A}_{\text{loop}}\left(\pi^0(p_a)\pi^0(p_b) \to \pi^1(p_c)\pi^1(p_d)\right) \sim \frac{p^4}{f_\pi^4}\int \frac{d^dk}{(2\pi)^d}\frac{1}{(k^2-m_\pi^2)^2} \sim \frac{p^4}{(4\pi f_\pi)^2 f_\pi^2}\left[\frac{1}{\epsilon} - \gamma_E + \ln 4\pi + \log\frac{\mu^2}{m_\pi^2}\right], \tag{7.47}$$

where I have been very sloppy with factors and signs.

The loop is divergent and gives a divergent contribution of the form p^4/ϵ. This of course has to be renormalized. But we have exactly the term that can do this if we look at the terms in Eq. (7.45). We can interpret the LECs L_i as the counter terms that will absorb the $1/\epsilon$ (and the associated μ dependence). Schematically again the total amplitude of the loop + NLO will look like

$$i\mathcal{A}_{NLO+\text{loop}}\left(\pi^0(p_a)\pi^0(p_b) \to \pi^1(p_c)\pi^1(p_d)\right) \sim \frac{p^4}{(4\pi f_\pi)^2 f_\pi^2}\left[\frac{1}{\epsilon} - \gamma_E + \ln 4\pi + \log\frac{\mu^2}{m_\pi^2}\right] + \frac{L_i p^4}{f_\pi^4}, \tag{7.48}$$

and if we want L_i to absorb the divergences we need $L_i \sim f_\pi^2/(4\pi f_\pi)^2$. We argued in the previous subsection that $L_i \sim f_\pi^2/\Lambda_\chi^2$ and from this we obtain the estimate of the χPT breakdown scale

$$\Lambda_\chi \simeq 4\pi f_\pi \simeq 1.2\,\text{GeV}. \tag{7.49}$$

Now we already saw that χPT only works if $p \ll \Lambda_\chi$ so we need p well below 1 GeV for χPT to work. People have worked very hard to also fit all the LECs L_i to data and, after appropriate renormalization, they for instance obtain $L_1 \simeq 4\cdot 10^{-3}$ which is not far away from our expectations $L_1 \simeq f_\pi^2/(4\pi f_\pi)^2 \simeq 6\cdot 10^{-3}$.

This discussions shows something interesting. In the EFT we studied before, such as the SM-EFT and LEFT, we had an expansion in $1/\Lambda$. In addition, there is the loop expansion in terms of the weak coupling constants such as $e^2/(4\pi)$. That is, at fixed order in $1/\Lambda$, we can compute loops up to any order. In χPT the expansion in $1/\Lambda_\chi$, however, is closely connected to the loop expansion. For instance, if we take only the LO Lagrangian then we cannot consider loops. Because the renormalization of the loops (with just LO vertices) requires an NLO interaction! In the same way if we include NLO interactions and consider 2-loop processes (with just LO vertices) of 1-loop processes (with an NLO vertex) we then need the next-to-next-to-leading order Lagrangian for renormalization.

In practice then going to higher-orders involves more and more LECs and, remember, they are a priori unknown. More and more data is needed to obtain all the LECs and at some point, in practice, predictive power is lost. However, already the first few orders of χPT are tremendously insightful. In modern applications χPT is extended to include baryon fields (such as nucleons) which allows the description of, for example, pion-nucleon scattering. χPT can also be extended to multi-nucleon systems and allows one to derive an EFT for nuclear forces. In this way it is possible to relate properties of large atomic nuclei more or less directly to QCD, with χPT as the stepping stone in between. Fascinating stuff!